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WEAKLY HYPERBOLIC EQUATIONS, SOBOLEV SPACES OF VARIABLE ORDER, AND PROPAGATION OF SINGULARITIES

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1. Introduction

Let us recall some results about linear and semilinear wave equations. We examine the Cauchy problems

$$\Box u = f(u), \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.1}$$

$$\Box v = 0, \qquad v(0,x) = u_0(x), \quad v_t(0,x) = u_1(x), \tag{1.2}$$

where $\Box = \partial_t^2 - \triangle$, $(t,x) \in \mathbb{R}_t \times \mathbb{R}_x^n$ and $f \in C^\infty$ with $f(0) = 0$. We suppose that the initial data u_0, u_1 satisfy $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s > n/2+1$. Then it is known that solutions u, v exist in $C([0,T], H^s) \cap C^1\left([0,T], H^{s-1}\right)$ for some small $T > 0$. Further, we assume that u_0, u_1 belong to C^∞ outside some closed set of $\mathbb{R}^n$. If singularities starting from two different points of this set of singularities meet, nothing happens in the linear case. They ignore each other and continue on their track. However, in the semilinear case, the nonlinear interaction of singularities may generate new singularities. These are weaker than those of v by at least one Sobolev order, which can be seen immediately as follows: we have $\Box(u - v) = f(u) \in C([0,T], H^s)$; hence $u - v \in C\left([0,T], H^{s+1}\right)$.

The aim of this publication is to prove a similar result for weakly hyperbolic equations whose lower order terms satisfy sharp Levi conditions. To demonstrate the phenomena which may occur in this setting, we recall a result of [15]. Let $v = v(t,x)$ be the solution of

$$v_{tt} - t^2 v_{xx} - b v_x = 0, \quad v(0,x) = u_0(x), \quad v_t(0,x) = 0, \quad x \in \mathbb{R}. \tag{1.3}$$

If $b = 4m+1$ and $m \in \mathbb{N}_0$, then the solution is given by

$$v(t,x) = \sum_{j=0}^{m} C_{jm} t^{2j} (\partial_x^j u_0)\left(x + \frac{t^2}{2}\right) \tag{1.4}$$

with some constants C_{jm}; and C_{mm} does not vanish. We observe two phenomena. The first is the *loss of regularity*: if $u_0 \in H^s$, then $v(t,.) \in H^{s-m}$. There is *no classical* *solution* for $m > s - 5/2$! The second is that singularities of the datum only propagate along one characteristic.

The loss of regularity makes the investigation of the semilinear problem

$$u_{tt} - t^2 u_{xx} - bu_x = f(u), \quad u(0,x) = u_0(x), \quad u_t(0,x) = 0 \tag{1.5}$$

difficult, because the standard iteration procedure and fixed point principles do not work. The existence of a solution u with $u(t,.) \in H^{s-M}$ for some number M can be proved by a modified iteration technique, see for instance [11], [16]. See also [5] for similar considerations in the C^∞ class. However, that method gives no information about the propagation of singularities, because only rough estimates of M can be found, which are generally not sharp.

If one is interested in the propagations of singularities in Sobolev spaces, then it is of great importance to know the spaces in which the description of singularities makes sense. To clarify this point, let us consider (1.1) and (1.2) with $u_0 \in H^s$, $u_1 \in H^{s-1}$. It is a true statement that, for instance, u and v belong to $C\left([0,T], H^{s-5}\right)$, while $u - v \in C\left([0,T], H^{s-4}\right)$. However, it makes no sense to investigate singularities in that space, because the singular support is the empty set. The right statement is $u, v \in C\left([0,T], H^s\right)$ and $u - v \in C\left([0,T], H^{s+1}\right)$, but, in general, $v \notin C\left([0,T], H^{s+1}\right)$.

Results of the following general type tell us that the strongest singularities of u and v coincide: we construct two function spaces $B_2 \subset B_1$, where the functions of B_2 have higher smoothness than that of B_1. Then we prove that $u, v \in B_1$ and $u - v \in B_2$. These statements are sharp in the sense that examples show that $v \notin B_2$.

The usual iteration procedure, even in its modified version mentioned above, is not able to give us a *sharp* description of the smoothness of v and u solving (1.3), (1.5). Another way to attack (1.5) consists in the construction of a special function space. This space contains all functions $w(t,x)$ with $\vartheta(t,\xi)\hat{w}(t,\xi) \in C([0,T], L^2)$, where $\vartheta(0,\xi) = \mathcal{O}(\langle\xi\rangle^s)$ and $\vartheta(t,\xi) = \mathcal{O}(\langle\xi\rangle^{s-m})$ $(t \neq 0)$ for $\langle\xi\rangle \to \infty$. Utilizing this idea, in [7] it was shown that the solution u of (1.5) belongs to $C\left([0,T], H^{s-m}\right)$ and that $u - v \in C\left([0,T], H^{s-m+1/2}\right)$. By (1.4), $v \notin C\left([0,T], H^{s-m+1/2}\right)$. In other words, also the strongest singularities of u propagate only along one characteristic. The idea to assign a weight $\vartheta(t,\xi)$ to the hyperbolic operator and to estimate a certain norm of the product $\vartheta(t,\xi)\hat{w}(t,\xi)$ goes back to [17]. The coefficients in the Cauchy problems of [7] and [17] did not depend on x. Therefore it was possible to apply partial Fourier transform and study the arising ODEs, in contrast to the situation in this publication, where the coefficients may depend on x, too.

Consider the model problems

$$\begin{aligned} u_{tt} - t^2 u_{xx} - (4m(x)+1)u_x &= f(u), \quad u(0,x) = u_0(x), \quad u_t(0,x) = 0, \\ v_{tt} - t^2 v_{xx} - (4m(x)+1)v_x &= 0, \quad v(0,x) = u_0(x), \quad v_t(0,x) = 0, \end{aligned}$$

where $m \in C_b^\infty(\mathbb{R})$ and $m \geq 0$. One expects a loss of $m(x)$ derivatives at the point

x. In Section 7, we will show that this variable loss of regularity happens indeed. As explained above, a reasonable description of singularities in spaces of Sobolev type requires to find the right spaces. Therefore we have to employ *Sobolev spaces with variable order of differentiation* $H^{s+m(x)}$:

$$H^{s+m(x)} := \{w \in L^2(\mathbb{R}) : \langle D_x \rangle^{s+m(x)} w(x) \in L^2(\mathbb{R})\}, \quad s \geq 0.$$

The main results of this publication (Theorems 4.1, 6.1 and 6.2) applied to these model problems yield:

Proposition 1.1. *Assume that $f = f(u)$ is entire analytic with $f(0) = 0$ and $u_0 \in H^{s+m(x)}$ with $s > 5/2$. Then some $T > 0$ exists with*

$$u,\ v \in C\left([0,T], H^s\right), \quad u - v \in C\left([0,T], H^{s+1/2}\right).$$

Examples show that the statement about the regularity of v is the best possible.

The Cauchy problems to be studied in this paper have the form

$$Lu = f(t,x,u), \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.6}$$

$$Lv = 0, \qquad v(0,x) = u_0(x), \quad v_t(0,x) = u_1(x), \tag{1.7}$$

where

$$L = D_t^2 + 2\sum_{j=1}^n t^{l_*} c_j(t,x) D_t D_{x_j} - \sum_{i,j=1}^n t^{2l_*} a_{ij}(t,x) D_{x_i} D_{x_j}$$
$$- \sum_{j=1}^n i l_* t^{l_*-1} b_j(t,x) D_{x_j} + i c_0(t,x) D_t, \quad l_* \in \mathbb{N}_+.$$

In [1] and [18], special linear model equations of this kind have been investigated and it was shown that the propagation of singularities depends in a sensitive way on lower order terms. Similarly to (1.3), sometimes the singularities propagate only in one direction. The special choice of t-exponents reflects the so-called Levi conditions which are necessary and sufficient conditions for the C^∞ well-posedness, compare [9] and [14]. For related results on propagation of C^∞ singularities, for example see [10].

Our assumptions are the following:

$$a_{ij}, c_j, b_j \in C_b^\infty([0,T_0] \times \mathbb{R}^n, \mathbb{R}), \tag{1.8}$$

$$\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j + \left(\sum_{j=1}^n c_j(t,x)\xi_j\right)^2 \geq \alpha_0|\xi|^2, \tag{1.9}$$
$$\alpha_0 > 0, \quad (t,x,\xi) \in [0,T_0] \times \mathbb{R}^{2n},$$

$$f(t,x,u)=\sum_{j=1}^{\infty} f_j(t,x)u^j, \quad f_j \in C_b^{\infty}([0,T_0]\times\mathbb{R}^n), \tag{1.10}$$

$$\sup_{[0,T_0]} \sum_{j=1}^{\infty} \|f_j(t,.)\|_{C_b^k} u^j < \infty \quad \forall\, k\in\mathbb{N}, \quad \forall\, u\in\mathbb{R}. \tag{1.11}$$

The paper is organised as follows. A theory of Sobolev spaces of variable order is presented in Section 2. We also list some properties of special symbol classes which are a main ingredient in the construction of parametrix in Section 3. This fundamental solution is utilized to prove the well-posedness of the linear Cauchy problem in Section 4. The investigation of the semilinear Cauchy problem in Section 6 relies on the algebra property of our function spaces proved in Section 5. Finally, we show that our results are sharp and discuss an application to propagation of singularities in Section 7.

2. Function spaces and symbol classes

For convenience, we introduce the notation $\lambda(t)=t^{l_*}$, $\Lambda(t)=\int_0^t\lambda(\tau)\,d\tau$. If X is some set from $\mathbb{R}^K$, then $C_b^m(X)$ denotes the space of all functions, whose derivatives up to the order m are continuous and bounded functions over X. Let $S^m_{\varrho,\delta}$ and $\Psi^m_{\varrho,\delta}$ denote the usual spaces of symbols and pseudodifferential operators, respectively,

$$S^m_{\varrho,\delta} := \{p(x,\xi)\in C^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_\xi): \quad \pi^m_{\alpha,\beta}(p)<\infty \quad \forall\,\alpha,\beta\in\mathbb{N}^n\},$$
$$\pi^m_{\alpha,\beta}(p) := \sup\left\{\left|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)\right|\langle\xi\rangle^{-m+\varrho|\beta|-\delta|\alpha|} : (x,\xi)\in\mathbb{R}^{2n}\right\},$$

where we employ the usual multi-index notation and $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. To simplify notation, we adjust the measure in the cotangent space, $d\xi=(2\pi)^{-n}d\xi_1\cdots d\xi_n$. If $p(x,\xi)$ is some symbol, the corresponding pseudodifferential operator will be denoted by the upper case letter, $P=P(x,D_x)$. If $P(x,D_x)$ and $Q(x,D_x)$ are pseudodifferential operators, then $P\cdot Q=(P\cdot D)(x,D_x)$ is defined as the operator with the symbol $(\mathrm{sym}(P\cdot Q))(x,\xi):=p(x,\xi)q(x,\xi)$. If $p(x,\xi)$ and $q(x,\xi)$ are symbols, the symbol $(p\circ q)(x,\xi)$ is defined by the asymptotic expansion

$$(p\circ q)(x,\xi)\sim\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}(D_\xi^\alpha p(x,\xi))(\partial_x^\alpha q(x,\xi)), \quad D_\xi:=-i\partial_\xi:=-i\frac{\partial}{\partial\xi}.$$

This symbol is unique modulo $S^{-\infty}$. Let us derive some auxiliary results, citing ideas from [8], Chapter 22.

DEFINITION 2.1. Let $\mathcal{K}_{\varrho,\delta}$ be the set of symbols

$$\mathcal{K}_{\varrho,\delta}=\left\{a\in\bigcup_{m\in\mathbb{R}}S^m_{\varrho,\delta}: \ \left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|\le C_{\alpha\beta}a(x,\xi)\langle\xi\rangle^{-\varrho|\beta|+\delta|\alpha|} \ \ \forall\xi,\alpha,\beta\right\}.$$

Proposition 2.2. *Let $a, a_0, a_1 \in \mathcal{K}_{\varrho,\delta}$. Then $a_0 \cdot a_1$, $a^r \in \mathcal{K}_{\varrho,\delta}$ for all $r \in \mathbb{R}$.*

Proof. The statement about $a_0 \cdot a_1$ follows from the Leibniz formula. Concerning a^r, we employ Faà di Bruno's formula

$$\partial_x^\alpha \partial_\xi^\beta G(a(x,\xi)) = \sum_{\nu=1}^{|\alpha|+|\beta|} G^{(\nu)}(a(x,\xi)) \sum_{(\alpha,\beta,\nu)} \prod_{\mu=1}^{\nu} \left(\partial_x^{\alpha^\mu} \partial_\xi^{\beta^\mu} a(x,\xi) C_{\alpha\beta\nu\mu}\right),$$
$$\sum_{(\alpha,\beta,\nu)} := \sum_{\substack{(\alpha^1,\beta^1)+\dots+(\alpha^\nu,\beta^\nu)=(\alpha,\beta),\\ |\alpha^\mu|+|\beta^\mu|>0}},$$

with $G(s) = s^r$ and the proof is complete. $\square$

2.1. Sobolev spaces of variable order The space $C([0,T], H^s)$ consists of all functions $w(t,x)$ with $\langle\xi\rangle^s \hat{w}(t,\xi) \in C([0,T], L^2(\mathbb{R}^n_\xi))$. We generalize this space, replacing $\langle D\rangle^s$ by some pseudodifferential operator $\Theta_M(t,x,D_x)$, whose symbol $\vartheta_M(t,x,\xi)$ may have different growth rates with respect to $\langle\xi\rangle$, depending on t and x. In the following, adapted symbol classes, embeddings in the usual Sobolev spaces (and vice versa), and mapping properties of operators from $\Psi^m_{\varrho,\delta}$ are described.

Let $\vartheta \in C([0,T], S^{K_+}_{1-\varepsilon,\varepsilon})$, $0 < \varepsilon < 1/2$, be a symbol such that

$$\vartheta(t,x,\xi) \geq C_1 \langle\xi\rangle^{K_-}, \quad C_1 > 0, \quad (t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}, \tag{2.1}$$
$$\vartheta(t,.,.) \in \mathcal{K}_{1-\varepsilon,\varepsilon}, \text{ uniformly in } t. \tag{2.2}$$

We choose some $M \in \mathbb{R}$ and define $\vartheta_M(t,x,\xi) := \vartheta(t,x,\xi)\langle\xi\rangle^M$. Let us describe the symbol ϑ more closely.

Proposition 2.3. *If $\vartheta \in C([0,T], S^{K_+}_{1-\varepsilon,\varepsilon})$ satisfies* (2.1), (2.2), *then*

$$\frac{D_x^\alpha D_\xi^\beta \vartheta(t,x,\xi)}{\vartheta(t,x,\xi)} \in S^{-(1-\varepsilon)|\beta|+\varepsilon|\alpha|}_{1-\varepsilon,\varepsilon}, \quad \alpha,\beta \in \mathbb{N}^n, \quad 0 < \varepsilon < \frac{1}{2}. \tag{2.3}$$

Proof. The assertion follows from the Leibniz rule and Proposition 2.2. $\square$

Proposition 2.4 ([8], Theorem 22.1.3). *For each ϑ as above, there are symbols $\vartheta^\sharp \in L^\infty([0,T], S^{-K_-}_{1-\varepsilon,\varepsilon})$ and $r_\infty \in L^\infty([0,T], S^{-\infty})$ with $\Theta^\sharp\Theta = I + R_\infty$.*

The usual symbol calculus and $\vartheta(t,.,.) \in \mathcal{K}_{1-\varepsilon,\varepsilon}$ lead to the following result.

Lemma 2.5. *For each* $a, \tilde{b} \in L^\infty([0,T], S^m_{1-\varepsilon,\varepsilon})$ *there are symbols* $\tilde{a}, b \in L^\infty([0,T], S^m_{1-\varepsilon,\varepsilon})$ *with the property that*

$$a\vartheta - \tilde{a} \circ \vartheta, \quad b\vartheta - \tilde{b} \circ \vartheta \in L^\infty([0,T], S^{-\infty}).$$

Similar statements hold, when ϑ *is the factor on the left.*

The class of these symbols ϑ_M is a special case of the fairly general symbol classes $S^\mu_{\Phi,\phi}$ of Beals and Fefferman ([2], [3], [4]) which consist of symbols a satisfying

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta} \exp(\mu(x,\xi))\phi(x,\xi)^{-|\alpha|}\Phi(x,\xi)^{-|\beta|}$$

with

$$\begin{aligned}
&\phi \le C, \quad \Phi\phi \ge c,\\
&c \le \Phi(x,\xi)\Phi(y,\eta)^{-1} \le C \text{ and } c \le \phi(x,\xi)\phi(y,\eta)^{-1} \le C\\
&\quad \text{if } |x-y| \le c\phi(x,\xi) \text{ and } |\xi-\eta| \le c\Phi(x,\xi),\\
&R(x,0) \le C\langle x\rangle^C, \text{ where } R = \Phi\phi^{-1},\\
&c \le R(x,\xi)R(y,\eta)^{-1} \le C\\
&\quad \text{if } |\xi-\eta| \le CR(x,\xi)^{\delta+1/2}, \quad |x-y| \le CR(x,\xi)^\delta R(y,\eta)^{-1/2}, \quad \delta > 0,\\
&|\mu(x,\xi) - \mu(y,\eta)| \le C \text{ if } |x-y| \le C\phi(x,\xi), \quad |\xi-\eta| \le \Phi(x,\xi),\\
&c(\Phi\phi)^{-m} \le e^\mu \Phi^{-K}\phi^{-k} \le C(\Phi\phi)^m \text{ for some } K, k, m.
\end{aligned}$$

In our case, $\exp\mu = \vartheta_M$, $\Phi = \langle\xi\rangle^{1-\varepsilon}$, $\phi = \langle\xi\rangle^{-\varepsilon}$. The results of [3] enable us to characterize Sobolev spaces of variable order, which we define now.

DEFINITION 2.6 (Sobolev spaces of variable order). If $M + K_- \ge 0$, then let $B_{\vartheta,M,T}$ be the space

$$B_{\vartheta,M,T} := \{u \in C([0,T], L^2) : \quad \Theta_M(t,x,D_x)u \in C([0,T], L^2)\}.$$

This space is called *Sobolev space of variable order* and has the norm

$$\|u\|_{M,T} := \sup_{[0,T]}(\|\Theta_M(t,x,D_x)u(t,x)\|_{L^2} + \|u(t,x)\|_{L^2}).$$

The next two propositions follow from Theorem 6.1 of [3].

Proposition 2.7. *The Banach spaces* $B_{\vartheta,M,T}$ *satisfy the embeddings*

$$C\left([0,T], H^{M+K_+}\right) \subset B_{\vartheta,M,T} \subset C\left([0,T], H^{M+K_-}\right).$$

Proposition 2.8. *If* $p \in C([0,T], S^m_{\varrho,\delta})$ *with* $\min(\varrho, 1-\varepsilon) > \max(\delta, \varepsilon)$*, then*

$$P : B_{\vartheta,M+m,T} \to B_{\vartheta,M,T} \text{ continuously.}$$

2.2. Symbol classes in $Z_{hyp}(N)$ We split the (t,ξ) space into two zones, the so-called *pseudodifferential zone* $Z_{pd}(N)$ and the *hyperbolic zone* $Z_{hyp}(N)$, where N is some large number:

$$\begin{aligned} Z_{pd}(N) &:= \{(t,\xi) : |\xi| \geq 1, \quad \Lambda(t)\langle\xi\rangle \leq N\}, \\ Z_{hyp}(N) &:= \{(t,\xi) : |\xi| \geq 1, \quad \Lambda(t)\langle\xi\rangle \geq N\}. \end{aligned}$$

Let $t_\xi \in \mathbb{R}_+$ be defined by $\Lambda(t_\xi)\langle\xi\rangle = N$.

In order to describe the t-dependence of the pseudodifferential operators referred to in this paper more precisely, we introduce the following classes of symbols in $Z_{hyp}(N)$. A detailed theory of these symbols can be found in [19].

DEFINITION 2.9. We say that $a(t,x,\xi) \in C^\infty([0,T] \times \mathbb{R}^{2n})$ belongs to the symbol class $S_N\{m_1, m_2, m_3\}$ if $\partial_t^j a \in C([0,T], S^{M_j}_{1,0})$ for all j and some M_j and

$$\left|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)\right| \leq C_{k\alpha\beta} \langle\xi\rangle^{m_1-|\beta|} \lambda(t)^{m_2} t^{m_3-k} \tag{2.4}$$

for all $k \geq 0$, $\alpha, \beta \in \mathbb{N}^n$, and all $(t,x,\xi) \in Z_{hyp}(N)$.

Proposition 2.10. *The symbols of these classes satisfy*

$$S_N\{m_1,m_2,m_3\} \subset S_N\{m_1+k, m_2+k, m_3+k\} \quad \forall\, k \geq 0, \tag{2.5}$$

$$a \in S_N\{m_1,m_2,m_3\} \Longrightarrow D_t^k D_\xi^\beta a \in S_N\{m_1-|\beta|, m_2, m_3-k\}, \tag{2.6}$$

$$\begin{aligned} &a \in S_N\{m_1,m_2,m_3\}, \quad b \in S_N\{k_1,k_2,k_3\} \\ &\qquad \Longrightarrow ab, \quad a \circ b \in S_N\{m_1+k_1, m_2+k_2, m_3+k_3\}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &a \in S_N\{m_1,m_2,m_3\}, \quad a(t,x,\xi) \equiv 0 \quad \textit{if} \quad \Lambda(t)\langle\xi\rangle \notin [N,N'] \\ &\qquad \Longrightarrow a \in S_N\{m_1-k, m_2-k, m_3-k\} \quad \forall\, k \geq 0. \end{aligned} \tag{2.8}$$

Proposition 2.11. (a) *Assume that* $a_k \in S_N\{m_{1k}, m_2, m_3\}$ $(k \in \mathbb{N}_0)$ *with* $m_{1k} \searrow -\infty$ *as* $k \to \infty$*. Suppose that each* a_k *vanishes in* $Z_{pd}(N)$*. Then there is a (unique up to* $C^\infty([0,T], S^{-\infty})$*) symbol* $a \in S_N\{m_{1(0)}, m_2, m_3\}$ *with support in* $Z_{hyp}(N)$ *and*

$$a - \sum_{j=0}^{k-1} a_j \in S_N\{m_{1k}, m_2, m_3\} \quad \forall\, k. \tag{2.9}$$

(b) *Suppose that* $b_k \in S_N\{m_1-k, m_2-k, m_3-k\}$ $(k \in \mathbb{N}_0)$*. Assume that each* b_k *vanishes in* $Z_{pd}(N)$*. Then there is a (unique up to* $\cap_{k\geq 0} S_N\{m_1-k, m_2-k, m_3-k\}$*)*

symbol $b \in S_N\{m_1, m_2, m_3\}$ *with support in* $Z_{hyp}(N)$ *and*

$$b - \sum_{j=0}^{k-1} b_j \in S_N\{m_1 - k, m_2 - k, m_3 - k\} \quad \forall\, k. \tag{2.10}$$

Sketch of proof. We choose some smooth function χ with $\chi(s) = 1$ for $|s| \le 1$, $\chi(s) = 0$ for $|s| \ge 2$ and set

$$a(t, x, \xi) := \sum_{k=0}^{\infty} (1 - \chi(\varepsilon_k \langle\xi\rangle)) a_k(t, x, \xi),$$
$$b(t, x, \xi) := \sum_{k=0}^{\infty} (1 - \chi(\delta_k \Lambda(t) \langle\xi\rangle)) b_k(t, x, \xi).$$

Here $\{\varepsilon_k\}$, $\{\delta_k\}$ are sequences of positive numbers, monotonically converging to zero. If we choose these numbers appropriately, then (2.9), (2.10) can be shown. For details, we refer the reader to [19, Proposition 3.3.2]. $\square$

Proposition 2.12 (Parametrix of elliptic operators). *Assume that the matrix symbol* $a \in S_N\{0, 0, 0\}$ *is constant in* $Z_{pd}(N)$ *and satisfies*

$$|\det a(t, x, \xi)| \ge \textit{ const } > 0$$

for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2n}$. *Then a parametrix* $B(t, x, D_x)$ *exists,*

$$BA - I, \quad AB - I \in C^\infty([0, T], \Psi^{-\infty}),$$

with the property that $b \in S_N\{0, 0, 0\}$ *and*

$$b(t, x, \xi) = a(t, x, \xi)^{-1} \quad \textit{ in } Z_{pd}(N).$$

Proof. We set $b_0(t, x, \xi) := a(t, x, \xi)^{-1}$ and observe that $b_0 \in S_N\{0, 0, 0\}$. Now we recursively define symbols b_k $(k \ge 1)$ by

$$\sum_{|\alpha|=1}^{k} \frac{1}{\alpha!} (D_\xi^\alpha a(t, x, \xi))(\partial_x^\alpha b_{k-|\alpha|}(t, x, \xi)) =: -a(t, x, \xi) b_k(t, x, \xi)$$

and see that $b_k(t, x, \xi) \equiv 0$ in $Z_{pd}(N)$ and $b_k \in S_N\{-k, 0, 0\}$. Proposition 2.11 gives us a symbol $b \in S_N\{0, 0, 0\}$ with

$$b - \sum_{j=0}^{k-1} b_j \in S_N\{-k, 0, 0\} \quad \forall\, k, \quad b(t, x, \xi) = a(t, x, \xi)^{-1} \text{ in } Z_{pd}(N).$$

By construction, we have $a \circ b - I \in C^\infty([0,T], S^{-\infty})$. The statement about the left parametrix can be proved in a similar way. $\square$

Finally, we connect the symbol class $S_N\{0,0,-1\}$ with the weight symbols from the class $\mathcal{K}_{\varrho,\delta}$.

Proposition 2.13. *If the symbol a vanishes in $Z_{pd}(N)$ and satisfies* (2.4) *with* $\{m_1, m_2, m_3\} = \{0,0,-1\}$ *and $k = 0$, then*

$$\exp\left(\int_0^t a(\tau,x,\xi)\,d\tau\right) \in \mathcal{K}_{1-\varepsilon,\varepsilon} \quad \forall\, \varepsilon > 0, \quad 0 \le t \le T,$$

$$\exp\left(\int_0^t a(\tau,x,\xi)\,d\tau\right) \in L^\infty([0,T], S^{K^+}_{1-\varepsilon,\varepsilon}) \quad \forall\, \varepsilon > 0$$

with $K_+ := \sup\{|a(t,x,\xi)\Lambda(t)/\lambda(t)| : (t,x,\xi) \in [0,T]\times\mathbb{R}^{2n}\}$.

Proof. We only note that $|\partial_\xi^\beta \int_0^t a(\tau,x,\xi)\,d\tau| \le C_\beta\langle\xi\rangle^{-|\beta|}\ln\langle\xi\rangle$, due to

$$\int_{t_\xi}^t \langle\xi\rangle^{-|\beta|}\frac{1}{\tau}d\tau = C\langle\xi\rangle^{-|\beta|}\ln\frac{\Lambda(t)}{\Lambda(t_\xi)} \le C\langle\xi\rangle^{-|\beta|}\ln\langle\xi\rangle. \qquad \square$$

In Section 4, we will fix some symbol $\tilde\beta$ which has, basically, the form

$$\tilde\beta(t,x,\xi) = \frac{\lambda(t)}{\Lambda(t)}\beta(x) \in S_N\{0,0,-1\}$$

in the hyperbolic zone. Then we will set (compare (4.2), (4.3))

$$\vartheta(0,x,\xi) := \exp\left(\int_{t_\xi}^T \tilde\beta(\tau,x,\xi)d\tau\right) = C(T,N,x)\langle\xi\rangle^{\beta(x)}.$$

This is a symbol whose growth rate will depend on x. That is the reason why we studied such weight symbols.

3. Fundamental solutions

The solution w to the Cauchy problem $Lw = f(t,x)$, $w(0,x) = u_0(x)$, $w_t(0,x) = u_1(x)$, behaves differently in the two zones $Z_{pd}(N)$ and $Z_{hyp}(N)$. In order to study fundamental solutions in each zone separately, we introduce the following cut functions. Fix some $\chi \in C^\infty(\mathbb{R})$ with $\chi(s) = 1$ for $s \le 1$, $\chi(s) = 0$ for $s \ge 2$, and $0 \le \chi(s) \le 1$ else. Then we set

$$\chi_N^+(t,\xi) := \chi\left(\frac{\Lambda(t)\langle\xi\rangle}{N}\right), \quad \chi_N^-(t,\xi) := 1 - \chi_N^+(t,\xi),$$

define the weight symbol

$$g(t,\xi) = \lambda(t)|\xi|\chi_N^-(t,\xi) + t_\xi^{-1}\chi_N^+(t,\xi), \tag{3.1}$$

and consider $W(t,x) = (G(t,D_x)w(t,x), D_t w(t,x))^T$. This vector solves the following pseudodifferential system of first order:

$$\begin{aligned} D_t W &= \begin{pmatrix} 0 & G \\ \sum \lambda^2 a_{ij} D_{x_i x_j} G^{-1} & -2\sum \lambda c_j D_{x_j} \end{pmatrix} W \\ &\quad + \begin{pmatrix} (D_t G)G^{-1} & 0 \\ -\sum (D_t\lambda) b_j D_{x_j} G^{-1} & 0 \end{pmatrix} W + \begin{pmatrix} 0 & 0 \\ 0 & -ic_0 \end{pmatrix} W + \begin{pmatrix} 0 \\ f \end{pmatrix} \\ &=: (\mathsf{A}_N^0 + \mathsf{A}_N^1 + \mathsf{A}_N^2)W + F =: \mathsf{A}_N W + F. \end{aligned}$$

For the further description of the symbols a_N^j, we define the set $\mathscr{H}(Z_{pd}(2N))$:

$$\begin{aligned} \mathscr{H}(Z_{pd}(2N)) := \Big\{ & p(t,x,\xi) \in C^\infty(Z_{pd}(2N)) : \\ & \left|\partial_x^\alpha \partial_\xi^\beta p(t,x,\xi)\right| \le C_{\alpha\beta} g(t,\xi)\langle\xi\rangle^{-|\beta|}, \quad (t,x,\xi) \in Z_{pd}(2N) \Big\}. \end{aligned}$$

We observe that

$$\begin{aligned} &\mathsf{a}_N^0, \mathsf{a}_N^1, \mathsf{a}_N^2 \in \mathscr{H}(Z_{pd}(2N)), \\ &\mathsf{a}_N^0 \in S_N\{1,1,0\}, \quad \mathsf{a}_N^1 \in S_N\{0,0,-1\}, \quad \mathsf{a}_N^2 \in S_N\{0,0,0\}. \end{aligned}$$

In the sequel, we consider the fundamental solution $\mathsf{E}(t,s)$, which satisfies

$$D_t \mathsf{E}(t,s) = \mathsf{A}_N(t)\mathsf{E}(t,s), \quad \mathsf{E}(s,s) = I, \quad 0 \le s, t \le T.$$

3.1. Diagonalization Let $\tau^\mp(t,x,\xi)$ be the characteristic roots of a_N^0 (in $Z_{hyp}(2N)$),

$$\begin{aligned} \tau^\mp(t,x,\xi) &:= \left(-c(t,x,\xi) \mp \sqrt{c(t,x,\xi)^2 + a(t,x,\xi)}\right)\lambda(t)|\xi|\chi_N^-(t,\xi) \\ &\quad \mp t_\xi^{-1}\chi_N^+(t,\xi), \end{aligned} \tag{3.2}$$

$$c(t,x,\xi) := \sum_{j=1}^n c_j(t,x)\frac{\xi_j}{|\xi|}, \quad a(t,x,\xi) := \sum_{i,j=1}^n a_{ij}(t,x)\frac{\xi_i\xi_j}{|\xi|^2}. \tag{3.3}$$

Then we set

$$\mathsf{m}(t,x,\xi) := \begin{pmatrix} 1 & \left((-c - \sqrt{c^2+a})\chi_N^-\right)(t,x,\xi) - \chi_N^+(t,\xi) \\ 1 & \left((-c + \sqrt{c^2+a})\chi_N^-\right)(t,x,\xi) + \chi_N^+(t,\xi) \end{pmatrix}^T.$$

From (1.9) it follows that $\det \mathsf{m} \geq \min(2, 2\sqrt{\alpha_0}) > 0$. We have $\mathsf{m} \in S_N\{0,0,0\}$ and the matrix m is constant in $Z_{pd}(N)$. Then Proposition 2.12 gives us a parametrix $\mathsf{M}^\sharp$ for the operator M. Since $\mathsf{a}_N \in S_N\{1,1,0\}$, we have

$$\mathsf{m}^\sharp \circ \mathsf{a}_N \circ \mathsf{m} = \mathsf{m}^\sharp \mathsf{a}_N \mathsf{m} + \mathsf{r}_1 + \mathsf{r}_\infty,$$

with remainders $\mathsf{r}_1 \in S_N\{0,1,0\}$, $\mathsf{r}_1 \equiv 0$ in $Z_{pd}(N)$ and $\mathsf{r}_\infty \in C^\infty([0,T], S^{-\infty})$. It remains to consider the product of symbols $\mathsf{m}^\sharp \mathsf{a}_N \mathsf{m}$. By the choice of m,

$$\mathsf{m}^\sharp \mathsf{a}_N^0 \mathsf{m} = \begin{cases} \operatorname{diag}(\tau^-, \tau^+) & : \Lambda(t)\langle\xi\rangle \geq 2N, \\ \in \mathscr{H}(Z_{pd}(2N)) & : \Lambda(t)\langle\xi\rangle \leq 2N. \end{cases}$$

We introduce the notation $\mathsf{d} := \operatorname{diag}(\tau^-, \tau^+)$. It can be shown that

$$\mathsf{m}^\sharp \mathsf{a}_N^1 \mathsf{m} = \begin{cases} \mathsf{h}^{(0)} & : \Lambda(t)\langle\xi\rangle \geq 2N, \\ \in \mathscr{H}(Z_{pd}(2N)) & : \Lambda(t)\langle\xi\rangle \leq 2N, \end{cases}$$

$$\mathsf{h}^{(0)}(t,x,\xi) = \frac{D_t\lambda(t)}{2\lambda(t)} \begin{pmatrix} 1 - \frac{b(t,x,\xi)+c(t,x,\xi)}{\sqrt{c(t,x,\xi)^2+a(t,x,\xi)}} & 1 - \frac{b(t,x,\xi)+c(t,x,\xi)}{\sqrt{c(t,x,\xi)^2+a(t,x,\xi)}} \\ 1 + \frac{b(t,x,\xi)+c(t,x,\xi)}{\sqrt{c(t,x,\xi)^2+a(t,x,\xi)}} & 1 + \frac{b(t,x,\xi)+c(t,x,\xi)}{\sqrt{c(t,x,\xi)^2+a(t,x,\xi)}} \end{pmatrix}$$
$$+ \mathsf{r}_1(t,x,\xi), \quad \mathsf{r}_1 \in S_N\{0,0,0\},$$

$$b(t,x,\xi) := -\sum_{j=1}^n b_j(t,x)\frac{\xi_j}{|\xi|}. \tag{3.4}$$

Finally, $\mathsf{m}^\sharp \mathsf{a}_N^2 \mathsf{m} \in S_N\{0,0,0\}$. In the sequel, R_∞ denotes a generic regularizing operator from $C^\infty([0,T], \Psi^{-\infty})$. Then we obtain

$$\begin{aligned} D_t(M^\sharp(t)\mathsf{E}) &= (D_t\mathsf{M}^\sharp)(\mathsf{M}\mathsf{M}^\sharp + \mathsf{R}_\infty)\mathsf{E} + \mathsf{M}^\sharp \mathsf{A}_N(\mathsf{M}\mathsf{M}^\sharp + \mathsf{R}_\infty)\mathsf{E} \\ &= (\mathsf{D} + \mathsf{H}^{(0)} \operatorname{Op}\chi_N^- + \tilde{\mathsf{H}}^{(1)} + \mathsf{R}_0)\mathsf{M}^\sharp\mathsf{E} + \mathsf{R}_\infty\mathsf{E}, \end{aligned}$$
$$\tilde{\mathsf{h}}^{(1)} \in S_N\{-1,-1,-2\}, \quad \mathsf{r}_0 \in \mathscr{H}(Z_{pd}(2N)),$$
$$\operatorname{supp}\tilde{\mathsf{h}}^{(1)} \subset Z_{hyp}(N), \quad \operatorname{supp}\mathsf{r}_0 \subset Z_{pd}(2N).$$

Here we have modified the term r_1 in the definition of $\mathsf{h}^{(0)}$ in order to include all terms from $S_N\{0,0,0\}$.

This was the first step of diagonalization. We shall apply further steps in order to diagonalize $\mathsf{h}^{(0)}$ and $\tilde{\mathsf{h}}^{(1)}$ modulo $\cap_p S_N\{-p,-p,-p-1\}$. For this purpose, we define

$$\mathsf{k}^{(0)} := \begin{pmatrix} \mathsf{h}_{11}^{(0)} & 0 \\ 0 & \mathsf{h}_{22}^{(0)} \end{pmatrix} \chi_N^- \in S_N\{0,0,-1\},$$
$$\mathsf{n}^{(1)} := \begin{pmatrix} 0 & \frac{\mathsf{h}_{12}^{(0)}}{\tau^+-\tau^-} \\ \frac{\mathsf{h}_{21}^{(0)}}{\tau^--\tau^+} & 0 \end{pmatrix} \chi_N^- \in S_N\{-1,-1,-1\}$$

and observe that $\mathsf{N}^{(1)}\mathsf{D} - \mathsf{D}\mathsf{N}^{(1)} = \mathsf{H}^{(0)} \operatorname{Op} \chi_N^- - \mathsf{K}^{(0)} + \mathsf{R}^{(0)}$ with remainder $\mathsf{r}^{(0)} \in S_N\{-1, 0, -1\} \subset S_N\{-1, -1, -2\}$. Then it can be concluded that

$$\begin{aligned}\mathsf{H}^{(1)} &:= (D_t - \mathsf{D} - \mathsf{H}^{(0)} \operatorname{Op} \chi_N^- - \tilde{\mathsf{H}}^{(1)})(I + \mathsf{N}^{(1)}) - (I + \mathsf{N}^{(1)})(D_t - \mathsf{D} - \mathsf{K}^{(0)}) \\ &= (D_t \mathsf{N}^{(1)}) + \mathsf{R}^{(0)} - \tilde{\mathsf{H}}^{(1)} - \mathsf{H}^{(0)} \operatorname{Op} \chi_N^- \mathsf{N}^{(1)} - \tilde{\mathsf{H}}^{(1)}\mathsf{N}^{(1)} + \mathsf{N}^{(1)}\mathsf{K}^{(0)};\end{aligned}$$

hence $\mathsf{h}^{(1)} \in S_N\{-1, -1, -2\}$. We see that $\mathsf{k}^{(0)} \equiv \mathsf{n}^{(1)} \equiv 0$ in $Z_{pd}(N)$. Modulo $C([0, T], S^{-\infty})$, we have $\mathsf{h}^{(1)} \equiv 0$ in $Z_{pd}(N)$. Inductively we set (for $\nu \geq 2$)

$$\mathsf{k}^{(\nu-1)} := \begin{pmatrix} \mathsf{h}_{11}^{(\nu-1)} & 0 \\ 0 & \mathsf{h}_{22}^{(\nu-1)} \end{pmatrix} \in S_N\{-\nu+1, -\nu+1, -\nu\},$$

$$\mathsf{n}^{(\nu)} := \begin{pmatrix} 0 & \frac{\mathsf{h}_{12}^{(\nu-1)}}{\tau^+ - \tau^-} \\ \frac{\mathsf{h}_{21}^{(\nu-1)}}{\tau^- - \tau^+} & 0 \end{pmatrix} \in S_N\{-\nu, -\nu, -\nu\},$$

$$\begin{aligned}\mathsf{H}^{(\nu)} := {} & \left(D_t - \mathsf{D} - \mathsf{H}^{(0)} \operatorname{Op} \chi_N^- - \tilde{\mathsf{H}}^{(1)}\right) \left(I + \sum_{\mu=1}^{\nu} \mathsf{N}^{(\mu)}\right) \\ & - \left(I + \sum_{\mu=1}^{\nu} \mathsf{N}^{(\mu)}\right) \left(D_t - \mathsf{D} - \sum_{\mu=0}^{\nu-1} \mathsf{K}^{(\mu)}\right).\end{aligned}$$

From the induction assumption and Proposition 2.10, it follows that $\mathsf{h}^{(\nu)} \in S_N\{-\nu, -\nu, -\nu-1\}$ and $\mathsf{h}^{(\nu)} \equiv 0$ in $Z_{pd}(N)$, modulo $C([0, T], S^{-\infty})$.

Employing Proposition 2.11, we find symbols n and $\tilde{\mathsf{k}}$ that satisfy

$$\begin{aligned}&\mathsf{n} \in S_N\{0, 0, 0\}, \quad \tilde{\mathsf{k}} \in S_N\{-1, -1, -2\}, \\ &\mathsf{n}(t, x, \xi) \equiv I, \quad \tilde{\mathsf{k}}(t, x, \xi) \equiv 0 \text{ in } Z_{pd}(N), \\ &\mathsf{n} \sim I + \sum_{\mu=1}^{\infty} \mathsf{n}^{(\mu)} \mod \bigcap_{p \geq 0} S_N\{-p, -p, -p\}, \\ &\tilde{\mathsf{k}} \sim \sum_{\mu=1}^{\infty} \mathsf{k}^{(\mu)} \mod \bigcap_{p \geq 0} S_N\{-p, -p, -p-1\}.\end{aligned}$$

Then the operator identity

$$(D_t - \mathsf{D} - \mathsf{H}^{(0)} \operatorname{Op} \chi_N^- - \tilde{\mathsf{H}}^{(1)})\mathsf{N} = \mathsf{N}(D_t - \mathsf{D} - \mathsf{K}^{(0)} - \tilde{\mathsf{K}}) + \mathsf{K}^{(\infty)}$$

holds, where $\mathsf{K}^{(\infty)}$ is an operator with full matrix symbol

$$\mathsf{k}^{(\infty)} \in \bigcap_{p \geq 0} S_N\{-p, -p, -p-1\} \cap \mathscr{H}(Z_{pd}(2N)).$$

To complete the diagonalization, we need a parametrix for the operator N. The matrix norm of its symbol satisfies

$$\|\mathsf{n}(t,x,\xi) - I\| \le \frac{C}{\Lambda(t)\langle\xi\rangle} \le \frac{1}{3}$$

if $\Lambda(t)\langle\xi\rangle \ge N_1$, where $N_1 \gg N$ is an appropriate number. We define

$$\tilde{\mathsf{n}}(t,x,\xi) := \mathsf{n}(t,x,\xi) - \chi^{+}_{N_1}(t,\xi)(\mathsf{n}(t,x,\xi) - I)$$

and observe that $\mathsf{n} - \tilde{\mathsf{n}} \in \cap_{p\ge 0} S_N\{-p,-p,-p\}$ and

$$\|\tilde{\mathsf{n}}(t,x,\xi) - I\| \le \frac{1}{3} \quad \forall\,(t,x,\xi). \tag{3.5}$$

Since n is uniquely determined only modulo $\cap_{p\ge 0} S_N\{-p,-p,-p\}$, we may drop the tilde and assume that n satisfies (3.5), too. Proposition 2.12 shows that a parametrix $\mathsf{N}^\sharp$ to the operator N exists, whose symbol belongs to $S_N\{0,0,0\}$ and coincides with the identity matrix in $Z_{pd}(N)$.

If we allow some modifications in the term $\mathsf{K}^{(\infty)}$, we can show

$$(D_t - \mathsf{D} - \mathsf{K}^{(0)} - \tilde{\mathsf{K}} - \mathsf{K}^{(\infty)})(\mathsf{N}^\sharp \mathsf{M}^\sharp \mathsf{E}) = \mathsf{R}_\infty \mathsf{E}. \tag{3.6}$$

Thus we have proved:

Proposition 3.1. *The fundamental solution* $\mathsf{E}(t,s)$ *to the operator* $D_t - \mathsf{A}_N(t)$ *satisfies* (3.6), *where* $\mathsf{N}^\sharp$ *and* $\mathsf{M}^\sharp$ *are elliptic operators with symbols from* $L^\infty([0,T], S^0_{1,0})$; *and* D, $\mathsf{K}^{(0)}$, $\tilde{\mathsf{K}}$ *are diagonal operators with symbols from* $S_N\{1,1,0\}$, $S_N\{0,0,-1\}$, $S_N\{-1,-1,-2\}$, *respectively. The* (*full matrix*) *symbol of* $\mathsf{K}^{(\infty)}$ *belongs to* $\cap_{p\ge 0} S_N\{-p,-p,-p-1\}$ *and* $\mathscr{H}(Z_{pd}(2N))$. *In* $Z_{pd}(N)$, *the symbol of* D *is independent of* (t,x) *and the symbols of* $\mathsf{K}^{(0)}$, $\tilde{\mathsf{K}}$ *vanish. Moreover, the symbols of* D *and* $\mathsf{K}^{(0)}$ *are given by*

$$\begin{aligned} &\mathsf{d} = \operatorname{diag}(\tau^-, \tau^+), \quad \mathsf{k}^{(0)} = \operatorname{diag}(\mathsf{k}^{(0)}_-, \mathsf{k}^{(0)}_+), \\ &\mathsf{k}^{(0)}_\mp = \frac{D_t\lambda}{2\lambda}\left(1 \mp \frac{b+c}{\sqrt{c^2+a}}\right)\chi^-_N + \mathsf{r}_\mp \chi^-_N, \quad \mathsf{r}_\mp \in S_N\{0,0,0\}, \end{aligned} \tag{3.7}$$

see (3.2), (3.3) *and* (3.4).

3.2. Construction of the fundamental solution Now we are going to construct $\mathsf{E}(t,s)$. Let P_2, P_1, P_0 denote the operators

$$\mathsf{P}_2 := \mathsf{D} + \mathsf{K}^{(0)} + \tilde{\mathsf{K}} + \mathsf{K}^{(\infty)}, \quad \mathsf{P}_1 := \mathsf{D} + \mathsf{K}^{(0)} + \tilde{\mathsf{K}}, \quad \mathsf{P}_0 := \mathsf{D}$$

and $\mathsf{E}_j(t,s)$, $(j = 0, 1, 2)$, the fundamental solutions to $D_t - \mathsf{P}_j$:

$$(D_t - \mathsf{P}_j(t))\mathsf{E}_j(t,s) = 0, \quad \mathsf{E}_j(s,s) = I.$$

Since $\mathsf{E}(t,s) = \mathsf{M}(t)\mathsf{N}(t)\mathsf{E}_2(t,s)\mathsf{N}^\sharp(s)\mathsf{M}^\sharp(s) + \mathsf{R}_\infty(t,s)$, it suffices to construct E_2. This task is done in three steps. First, we construct E_0, which is a diagonal Fourier integral operator of order zero. Then we compute $\mathsf{E}_1 = \mathsf{E}_0\mathsf{Q}_0$ by means of Egorov's Theorem. It turns out that Q_0 is a diagonal pseudodifferential operator which describes the loss of regularity. Finally, $\mathsf{E}_2 = \mathsf{E}_1\mathsf{Q}_1$, where Q_1 is some matrix pseudodifferential operator of order zero.

Since these constructions are quite standard, we only sketch them. For an exhaustive representation and related problems, see [19].

3.2.1. Fundamental solutions to scalar first order operators

DEFINITION 3.2. Let $\tau = \tau(t,x,\xi)$ be either τ^- or τ^+. The Hamilton flow $\mathscr{H}_{s,t}(y,\eta) := (x,\xi) = (x,\xi)(t,s,y,\eta)$ is defined as the solution to the system of ODEs

$$\begin{aligned} \frac{dx}{dt} &= -\nabla_\xi \tau(t,x,\xi), \quad x(s,s,y,\eta) = y, \\ \frac{d\xi}{dt} &= \nabla_x \tau(t,x,\xi), \quad \xi(s,s,y,\eta) = \eta. \end{aligned}$$

It is known that for small T, $0 < T \le T_0$, the solution (x,ξ) exists (uniformly with respect to (y,η)) for $0 \le s,t \le T$ and that

$$\begin{aligned} &\left\{ \frac{x(t,s) - y}{t-s},\ \partial_t x(t,s),\ \partial_s x(t,s) \right\}_{0 \le s,t \le T} \quad \text{is bounded in } S^0_{1,0}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta), \\ &\left\{ \frac{\xi(t,s) - \eta}{t-s},\ \partial_t \xi(t,s),\ \partial_s \xi(t,s) \right\}_{0 \le s,t \le T} \quad \text{is bounded in } S^1_{1,0}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta). \end{aligned}$$

If T is sufficiently small, then an inverse function $y = y(t,s,x,\eta)$ to the mapping $x = x(t,s,y,\eta)$ can be found. Then the set

$$\left\{ \frac{y(t,s) - x}{t-s},\ \partial_t y(t,s),\ \partial_s y(t,s) \right\}_{0 \le s,t \le T}$$

is bounded in $S^0_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\eta)$.

Now we can construct the phase function ϕ which solves the eikonal equation

$$\partial_t \phi(t,x;s,\xi) - \tau(t,x,\nabla_x \phi(t,x;s,\xi)) = 0$$

with the initial condition $\phi(s, x; s, \xi) = x \cdot \xi$. We set

$$v(t, y, s, \eta) = y \cdot \eta - \int_s^t (\xi \cdot \nabla_\xi \tau - \tau)(\sigma, x(\sigma, s, y, \eta), \xi(\sigma, s, y, \eta))\, d\sigma$$

and can express the phase function in the form

$$\phi(t, x_0; s, \xi_0) := v(t, y(t, s, x_0, \xi_0), s, \xi_0).$$

The representation of v and the properties of $y(t, s, x, \xi)$ imply

$$\left|\partial_x^\alpha \partial_\xi^\beta (\phi(t, x; s, \xi) - x \cdot \xi)\right| \leq C_{\alpha\beta} \langle \xi \rangle^{1-|\beta|} \Lambda(s \vee t), \quad s \vee t \geq t_\xi, \tag{3.8}$$

where $s \vee t := \max(s, t)$. The function $\tau(t, x, \xi)$ does not depend on (t, x) if $(t, x, \xi) \in Z_{pd}(N)$. Consequently,

$$\phi(t, x; s, \xi) = x \cdot \xi - (t - s)\tau(\xi), \quad s, t \leq t_\xi. \tag{3.9}$$

In order to formulate the transport equations for the amplitude functions of E_0, we recall a theorem about compositions of pseudodifferential operators and Fourier integral operators, see e.g., [12].

Theorem 3.3. *Let P_1 be a pseudodifferential operator with symbol $p_1(x, \xi) \in S^{m_1}_{1,0}$ and let $P_{2\phi}$ be a Fourier integral operator with phase function $\phi(x, \xi)$ and amplitude $p_2(x, \xi) \in S^{m_2}_{1,0}$. We assume that $\pi^1_{\alpha,\beta}(\phi(x, \xi) - x \cdot \xi) \leq \delta < 1$ for $|\alpha| + |\beta| \leq 2$. Then the composition $P_1 P_{2\phi}$ is a Fourier integral operator with phase function ϕ and amplitude $r(x, \xi) \in S^{m_1+m_2}_{1,0}$ which can be written as*

$$\begin{aligned} r(x, \xi) =& p_1(x, \nabla_x \phi(x, \xi)) p_2(x, \xi) + \sum_{j=1}^n (\partial_{\xi_j} p_1)(x, \nabla_x \phi(x, \xi)) D_{x_j} p_2(x, \xi) \\ & - \frac{i}{2} \sum_{j,k=1}^n \left(\partial^2_{\xi_j \xi_k} p_1\right)(x, \nabla_x \phi(x, \xi)) \left(\partial^2_{x_j x_k} \phi(x, \xi)\right) p_2(x, \xi) + r_2(x, \xi), \end{aligned}$$

with some $r_2 \in S^{m_1+m_2-2}_{1,0}$. We have the asymptotic expansion

$$r(x, \xi) \sim \sum_{|\alpha|=0}^\infty \frac{1}{\alpha!} D_y^\alpha \left((\partial_\xi^\alpha p_1)(x, \tilde{\nabla}_x \phi(x, y, \xi)) p_2(y, \xi)\right)_{y=x},$$

where $\tilde{\nabla}_x \phi(x, y, \xi) := \int_0^1 (\nabla_x \phi)(y + s(x - y), \xi)\, ds$.

This theorem is now employed to find the fundamental solution to $D_t - \mathsf{D}$.

Proposition 3.4. *The fundamental solution* $\mathsf{E}_0(t,s)$ *to the operator* $D_t - \mathsf{D}$ *is a diagonal Fourier integral operator,*

$$\begin{aligned}
&\mathsf{E}_0(t,s) = \operatorname{diag}(\mathsf{E}_0^-(t,s), \mathsf{E}_0^+(t,s)),\\
&(\mathsf{E}_0^{\mp}(t,s)w)(x) = \int e^{i\phi^{\mp}(t,x;s,\xi)} e^{\mp}(t,x;s,\xi)\hat{w}(\xi)\,d\xi,\\
&\phi^{\mp}(s,x;s,\xi) = x\cdot\xi, \quad e^{\mp}(s,x;s,\xi) = 1.
\end{aligned}$$

The phase functions ϕ^+, ϕ^- *satisfy* (3.8), (3.9), *and the amplitude functions* $e^{\mp}(t,x;s,\xi)$ *belong to* $C([0,T]^2, S^0_{1,0})$.

Proof. We shall express $e^{\mp}$ as an asymptotic series,

$$e^{\mp}(t,x;s,\xi) \sim \sum_{j=0}^{\infty} e_j^{\mp}(t,x;s,\xi), \quad e_j^{\mp} \in C([0,T]^2, S^{-j}_{1,0}),$$

with the initial conditions $e_j^{\mp}(s,x;s,\xi) = 0$ for $j \geq 1$. Obviously,

$$(D_t \mathsf{E}_0^{\mp}(t,s)w)(x) = \int e^{i\phi^{\mp}(t,x;s,\xi)}((\phi_t^{\mp}e^{\mp}) + i^{-1}e_t^{\mp})(t,x;s,\xi)\hat{w}(\xi)\,d\xi.$$

Let $\mathsf{E}_{0m}^{\mp}(t,s)$ denote the Fourier integral operator with phase function $\phi^{\mp}(t,x;s,\xi)$ and amplitude $e_m^{\mp}(t,x;s,\xi)$. Theorem 3.3 yields

$$\begin{aligned}
&(\tau^{\mp}(t,x,D_x)\mathsf{E}_{0m}^{\mp}(t,s)w)(x) =\\
&\quad\int e^{i\phi^{\mp}(t,x;s,\xi)}\left\{\tau^{\mp}(t,x,\nabla_x\phi^{\mp}(t,x;s,\xi))e_m^{\mp}(t,x;s,\xi)\right.\\
&\qquad + i^{-1}(\nabla_\xi\tau^{\mp})(t,x,\nabla_x\phi^{\mp}(t,x;s,\xi))\nabla_x e_m^{\mp}(t,x;s,\xi)\\
&\qquad - \frac{i}{2}\sum_{k,l}\left(\partial^2_{\xi_k\xi_l}\tau^{\mp}\right)(t,x,\nabla_x\phi^{\mp}(t,x;s,\xi))\left(\left(\partial^2_{x_kx_l}\phi^{\mp}\right)e_m^{\mp}\right)(t,x;s,\xi)\\
&\qquad \left. + r_{m,2}(t,x;s,\xi)\right\}\hat{w}(\xi)\,d\xi, \qquad \operatorname{ord}(r_{m,2}) = -1-m.
\end{aligned}$$

This leads us to the eikonal equation

$$\phi_t^{\mp}(t,x;s,\xi) - \tau^{\mp}(t,x,\nabla_x\phi^{\mp}(t,x;s,\xi)) = 0$$

and the transport equations ($m = 0, 1, 2, \dots$)

$$\begin{aligned}
(3.10)\quad &(\partial_t - (\nabla_\xi\tau^{\mp})(t,x,\nabla_x\phi^{\mp}(t,x;s,\xi))\nabla_x)e_m^{\mp}(t,x;s,\xi)\\
&- \frac{1}{2}\sum_{k,l}\left(\partial^2_{\xi_k\xi_l}\tau^{\mp}\right)(t,x,\nabla_x\phi^{\mp}(t,x;s,\xi))\left(\partial^2_{x_kx_l}\phi^{\mp}(t,x;s,\xi)\right)e_m^{\mp}(t,x;s,\xi)
\end{aligned}$$

$$= \sum_{k=0}^{m-1} \sum_{|\alpha|=m+1-k} \frac{i}{\alpha!} D_y^\alpha \left(\left(\partial_\xi^\alpha \tau^\mp \right)(t, x, \tilde{\nabla}_x \phi^\mp(t, x, y; s, \xi)) e_k^\mp(t, y; s, \xi) \right)_{y=x}.$$

To solve the transport equations, we recall a well-known result from the theory of first order PDEs. Consider the Cauchy problem with parameter p,

$$\left(\partial_t - \sum_{j=1}^n a_j(t, x; p) \partial_{x_j} \right) w(t, x, s; p) - a_0(t, x; p) w(t, x, s; p) = f(t, x; p),$$
$$w(s, x, s; p) = w_0(x; p).$$

Let $\gamma = \gamma(t, s, x_0; p) : [0, T]^2 \times \mathbb{R}^n \times \mathbb{R}_p^k \to \mathbb{R}^n$ be the solution to

$$\partial_t \gamma_j(t, s, x_0; p) = -a_j(t, \gamma(t, s, x_0; p); p), \quad j = 1, \dots, n,$$
$$\gamma(s, s, x_0; p) = x_0.$$

Then the solution w satisfies

$$w(t, x, s; p) = w_0(\gamma(s, t, x; p); p) \exp\left(\int_s^t a_0(r, \gamma(r, t, x; p); p)\, dr \right) + \int_s^t \exp\left(\int_\sigma^t a_0(r, \gamma(r, t, x; p); p)\, dr \right) f(\sigma, \gamma(\sigma, t, x; p); p)\, d\sigma. \tag{3.11}$$

In our situation, $p = (s, \xi)$, $a_i(t, x; p) = a_i^\mp(t, x; p)$, $(0 \le i \le n)$, with

$$a_i^\mp(t, x; p) = \left(\partial_{\xi_i} \tau^\mp \right)(t, x, \nabla_x \phi^\mp(t, x; s, \xi)), \quad 1 \le i \le n,$$
$$a_0^\mp(t, x; p) = \frac{1}{2} \sum_{k,l} \left(\partial_{\xi_k \xi_l}^2 \tau^\mp \right) \left(t, x, \nabla_x \phi^\mp(t, x; s, \xi) \right) \partial_{x_k x_l}^2 \phi^\mp(t, x; s, \xi),$$

and $f = f_m^\mp(t, x; p)$ is given by the right side of (3.10). Consequently,

$$e_0^\mp(t, x; s, \xi) = \exp\left(\int_s^t a_0^\mp(r, \gamma(r, t, x; s, \xi); s, \xi)\, dr \right),$$
$$e_m^\mp(t, x; s, \xi)$$
$$= \int_s^t \exp\left(\int_\sigma^t a_0^\mp(r, \gamma(r, t, x; s, \xi); s, \xi) dr \right) f_m^\mp(\sigma, \gamma(\sigma, t, x; s, \xi); s, \xi)\, d\sigma.$$

The coefficients a_i, $(0 \le i \le n)$, belong to $C([0, T]^2, S_{1,0}^0)$; more precisely,

$$\left| \partial_x^\alpha \partial_\xi^\beta a_i^\mp(t, x; s, \xi) \right| \le \begin{cases} C_{\alpha\beta} \lambda(t) \langle \xi \rangle^{-|\beta|} & : t_\xi \le t, \\ C_{\alpha\beta} t_\xi^{-1} \langle \xi \rangle^{-|\beta|-1} & : t \le t_\xi, \end{cases} \quad 1 \le i \le n,$$

$$\left|\partial_x^\alpha \partial_\xi^\beta a_0^\mp(t,x;s,\xi)\right| \le \begin{cases} C_{\alpha\beta}\lambda(t)\langle\xi\rangle^{-|\beta|}\Lambda(t\vee s) & : t_\xi \le t, \\ C_{\alpha\beta}t_\xi^{-1}\langle\xi\rangle^{-|\beta|-1}\Lambda(s) & : t \le t_\xi \le s, \\ 0 & : t,s \le t_\xi, \end{cases}$$

see (3.8), (3.9). Then we obtain $\gamma \in C([0,T]^3, S^0_{1,0})$ with $|\gamma(r,t,x;s,\xi)-x| \le C|r-t|$. The representation of $e_0^\mp$ implies $e_0^\mp(t,x;s,\xi) = 1$ if $s \vee t \le t_\xi$ and

$$\left|\partial_x^\alpha \partial_\xi^\beta e_0^\mp(t,x;s,\xi)\right| \le C_{\alpha\beta}\langle\xi\rangle^{-|\beta|}$$

if $s \vee t \ge t_\xi$. Inductively we conclude that

$$\left|\partial_x^\alpha \partial_\xi^\beta f_m^\mp(t,x;s,\xi)\right| \le \begin{cases} C_{\alpha\beta m}\lambda(t)\langle\xi\rangle^{-m-|\beta|} & : t_\xi \le t, \\ C_{\alpha\beta m}t_\xi^{-1}\langle\xi\rangle^{-m-1-|\beta|} & : t \le t_\xi \le s, \\ 0 & : t,s \le t_\xi \end{cases}$$

and

$$\left|\partial_x^\alpha \partial_\xi^\beta e_m^\mp(t,x;s,\xi)\right| \le \begin{cases} C_{\alpha\beta m}\Lambda(t)\langle\xi\rangle^{-m-|\beta|} & : t_\xi \le t, \\ C_{\alpha\beta m}\Lambda(s)\langle\xi\rangle^{-m-|\beta|} & : t \le t_\xi \le s, \\ 0 & : t,s \le t_\xi. \end{cases}$$

We see that $e_m^\mp \in C([0,T]^2, S^{-m}_{1,0})$. If we set

$$\tilde{e}^\mp(t,x;s,\xi) = e_0^\mp(t,x;s,\xi) + \sum_{j=1}^\infty (1-\chi(\varepsilon_j\langle\xi\rangle))e_j^\mp(t,x;s,\xi)$$

and choose the sequence $\{\varepsilon_j\}$ with $\varepsilon_j \searrow 0$ suitably, then

$$\tilde{e}^\mp(t,x;s,\xi) \sim \sum_{j=0}^\infty e_j^\mp(t,x;s,\xi) \quad \mod C([0,T]^2, S^{-\infty})$$

and $\tilde{e}^\mp(t,x;s,\xi) = 1$ for $s \vee t \le t_\xi$. The function $\tilde{e}$ is the amplitude to an approximate fundamental solution $\tilde{\mathsf{E}}_0^\mp$ which satisfies

$$(D_t - \tau^\mp(t,x,D_x))\tilde{\mathsf{E}}_0^\mp(t,x;s,D_x) = R_\infty^\mp(t,x;s,D_x)$$

with $r_\infty^\mp \in C([0,T]^2, S^{-\infty})$. In order to find the exact solution $\mathsf{E}_0^\mp$, we set

$$W_1(t,s) := -iR_\infty^\mp(t,s), \quad W_{\nu+1}(t,s) := \int_s^t W_1(t,\sigma)W_\nu(\sigma,s)\,d\sigma.$$

Then the desired fundamental solution to $D_t - \mathsf{D}$ is given by

$$\mathsf{E}_0^{\mp}(t,s) := \tilde{\mathsf{E}}_0^{\mp}(t,s) + \int_s^t \tilde{\mathsf{E}}_0^{\mp}(t,\sigma) \sum_{\nu=1}^{\infty} W_\nu(\sigma,s)\, d\sigma. \qquad \square$$

Now we construct $\mathsf{E}_1(t,s)$.

Proposition 3.5. *The fundamental solution* $\mathsf{E}_1(t,s)$ *to the operator* $D_t - \mathsf{D} - \mathsf{K}^{(0)} - \tilde{\mathsf{K}}$ *can be written as* $\mathsf{E}_1(t,s) = \mathsf{E}_0(t,s)\mathsf{Q}_0(t,s)$, *where* Q_0 *is a diagonal pseudodifferential operator,* $\mathsf{Q}_0 \in L^\infty([0,T]^2, S^{K_0}_{1-\varepsilon,\varepsilon})$ *for some* K_0. *With some* $\tilde{\mathsf{q}}_0 \in L^\infty([0,T]^2, S^0_{1-\varepsilon,\varepsilon})$, *its symbol can be written in the form*

$$\mathsf{q}_0(t,s,x,\xi) = \exp\left(\int_s^t i\mathsf{k}^{(0)}(\tau,s,x,\xi)\, d\tau\right) \tilde{\mathsf{q}}_0(t,s,x,\xi),$$
$$\mathsf{k}^{(0)}(t,s,x,\xi) = \operatorname{diag}\left(\mathsf{k}_-^{(0)}\left(t,\mathscr{H}^-_{s,t}(x,\xi)\right), \mathsf{k}_+^{(0)}\left(t,\mathscr{H}^+_{s,t}(x,\xi)\right)\right),$$

where the Hamilton flows $\mathscr{H}^{\mp}_{s,t}(x,\xi)$ *have been defined in* Definition 3.2; *and* $\mathsf{k}_\pm^{(0)}$ *is given by* (3.7).

Proof. We look for E_1 having the form $\mathsf{E}_1 = \mathsf{E}_0\mathsf{Q}_0$ and obtain

$$D_t\mathsf{Q}_0(t,s) = \mathsf{E}_0(s,t)(\mathsf{K}^{(0)}(t) + \tilde{\mathsf{K}}(t))\mathsf{E}_0(t,s)\mathsf{Q}_0(t,s).$$

According to Egorov's Theorem (see [19]), $\mathsf{E}_0(s,t)(\mathsf{K}^{(0)}(t)+\tilde{\mathsf{K}}(t))\mathsf{E}_0(t,s)$ is a pseudodifferential operator $\mathsf{K}(t,s)$, whose diagonal principal symbol $\mathsf{k}^{(0)}(t,s)+\mathsf{k}^{(1)}(t,s)$ is given by

$$\mathsf{k}_\mp^{(0)}(t,x;s,\xi) + \mathsf{k}_\mp^{(1)}(t,x;s,\xi) = \mathsf{k}_\mp^{(0)}(t,\mathscr{H}^{\mp}_{s,t}(x,\xi)) + \tilde{\mathsf{k}}_\mp(t,\mathscr{H}^{\mp}_{s,t}(x,\xi)),$$

see also Definition 3.2. Therefore, we write

$$\operatorname{sym}(\mathsf{K}(t,s)) =: \mathsf{k}(t,s) =: \mathsf{k}^{(0)}(t,s) + \mathsf{k}^{(1)}(t,s) + \mathsf{k}^{(2)}(t,s)$$

with lower order terms $\mathsf{k}^{(2)}(t,s)$ and look for Q_0 having the form

$$(\mathsf{Q}_0(t,s)w)(x) = \int e^{ix\xi}\mathsf{q}_0(t,s,x,\xi)\hat{w}(\xi)\, d\xi, \quad \mathsf{q}_0(s,s,x,\xi) = I.$$

We expand q_0 into an asymptotic series $\sum_{j=0}^\infty \mathsf{q}_{0j}$ with $\operatorname{ord}(\mathsf{q}_{0j}) = -j$ and

$$D_t\mathsf{q}_{0m}(t,s) = \sum_{|\alpha|=0}^{m} \frac{1}{\alpha!}(D_\xi^\alpha \mathsf{k}(t,s))(\partial_x^\alpha \mathsf{q}_{0(m-|\alpha|)}(t,s)).$$

Recalling that all matrices are diagonal, we see that

$$\mathsf{q}_{00}(t,s) = \exp\left(\int_s^t i\mathsf{k}(\tau,s)\,d\tau\right),$$
$$\mathsf{q}_{0m}(t,s) = i\int_s^t \exp\left(\int_\sigma^t i\mathsf{k}(\tau,s)\,d\tau\right)\sum_{|\alpha|=1}^{m}\frac{1}{\alpha!}\left(D_\xi^\alpha \mathsf{k}\right)\left(\partial_x^\alpha \mathsf{q}_{0(m-|\alpha|)}\right)(\sigma,s)d\sigma.$$

We know that the $\mathsf{k}^{(j)}$ vanish in $Z_{pd}(N)$ and that

$$\mathsf{k}^{(0)} \in \frac{\lambda}{\Lambda}C([0,T],S^0_{1,0}), \quad \mathsf{k}^{(1)} \in \frac{\lambda}{\Lambda^2}C([0,T],S^{-1}_{1,0}),$$
$$\mathsf{k}^{(2)} \in \frac{\lambda}{\Lambda^2}C([0,T],S^{-2}_{1,0}) \subset C([0,T],S^0_{1,0}),$$

uniformly with respect to $s\in[0,T]$. Consequently,

$$\int_{s\wedge t}^{s\vee t}\left|\partial_x^\alpha\partial_\xi^\beta \mathsf{k}^{(j)}(\tau,s,x,\xi)\right|\,d\tau \le C_{\alpha\beta}\langle\xi\rangle^{-|\beta|}, \quad j=1,2.$$

The proof of Proposition 2.13 gives the rough estimate

$$\int_{s\wedge t}^{s\vee t}\left|\partial_x^\alpha\partial_\xi^\beta \mathsf{k}^{(0)}(\tau,s,x,\xi)\right|\,d\tau \le C_{\alpha\beta\varepsilon}\langle\xi\rangle^{-|\beta|+\varepsilon} \quad \forall\,(t,s,x,\xi)$$

for every positive ε. By induction we obtain

$$\begin{aligned}&\left|\partial_x^\alpha\partial_\xi^\beta \mathsf{q}_{0m}(t,s,x,\xi)\right|\left\|\exp\left(-\int_s^t i\mathsf{k}^{(0)}(\tau,s,x,\xi)\,d\tau\right)\right\|\\&\quad\le C_{\alpha\beta\varepsilon m}\langle\xi\rangle^{-(1-\varepsilon)m+\varepsilon|\alpha|-(1-\varepsilon)|\beta|}.\end{aligned}$$

There is a positive real number K_+ with

$$\exp\left(\int_s^t \left\|\mathsf{k}^{(0)}(\tau,s,x,\xi)\right\|\,d\tau\right) \le \begin{cases} C\left(\frac{\Lambda(t\vee s)}{\Lambda(t\wedge s)}\right)^{K_+} & : t'_\xi \le s,t,\\ C\left(\frac{\Lambda(t\vee s)}{\Lambda(t_\xi)}\right)^{K_+} & : s\wedge t\le t'_\xi\le s\vee t,\\ 1 & : t,s\le t'_\xi.\end{cases}$$

Here t'_ξ is given by $\Lambda(t'_\xi)\langle\xi\rangle = N/2$. We shall see that the number K_+, which gives a bound for the loss of regularity, is the same number as in Section 2.1. Since $\mathsf{q}_{0m}\in L^\infty([0,T]^2,S^{K_+-(1-\varepsilon)m}_{1-\varepsilon,\varepsilon})$, the series $\sum \mathsf{q}_{0m}$ defines in a canonical way the symbol $\tilde{\mathsf{q}}_0$ of an approximative solution $\tilde{\mathsf{Q}}$. The solution Q_1 can be constructed from $\tilde{\mathsf{Q}}$ in the same way as in the proof of Proposition 3.4. □

3.2.2. Fundamental solutions to matrix first order operators Finally, we consider E_2. If $\mathsf{E}_2(t,s) = \mathsf{E}_1(t,s)\mathsf{Q}_1(t,s)$, then

$$D_t\mathsf{Q}_1(t,s) = \mathsf{E}_1(s,t)\mathsf{K}^{(\infty)}(t)\mathsf{E}_1(t,s)\mathsf{Q}_1(t,s), \quad \mathsf{Q}_1(s,s) = I. \tag{3.12}$$

The following proposition describes the factor on the right-hand side.

Proposition 3.6. *The operator* $\mathsf{K}^{(\infty)}(t,s) := \mathsf{E}_1(s,t)\mathsf{K}^{(\infty)}(t)\mathsf{E}_1(t,s)$, $s \le t$, *is a pseudodifferential operator whose* (*matrix*) *symbol satisfies, for every* $p \ge 0$,

$$\begin{aligned} &\left\|\partial_x^\alpha\partial_\xi^\beta \mathsf{k}^{(\infty)}(t,s,x,\xi)\right\| \\ &\quad\le \begin{cases} C_{\alpha\beta\varepsilon p}\left(\frac{\Lambda(t)}{\Lambda(s)}\right)^{2K_+}\frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : t'_\xi \le s \le t, \\ C_{\alpha\beta\varepsilon p}\left(\frac{\Lambda(t)}{\Lambda(t_\xi)}\right)^{2K_+}\frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : s \le t'_\xi \le t, \\ C_{\alpha\beta\varepsilon}t_\xi^{-1}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : s \le t \le t'_\xi. \end{cases} \end{aligned} \tag{3.13}$$

Proof. We start with the composition $\mathsf{K}^{(\infty)}(t)\mathsf{E}_1(t,s)$, which can be written as

$$\begin{aligned}(\mathsf{K}^{(\infty)}(t)\mathsf{E}_1(t,s)W)(x) = &\int e^{i\phi^-(t,s,x,\xi)}\mathsf{k}^{1-}(t,s,x,\xi)\hat{W}(\xi)\,d\xi \\ &+ \int e^{i\phi^+(t,s,x,\xi)}\mathsf{k}^{1+}(t,s,x,\xi)\hat{W}(\xi)\,d\xi.\end{aligned}$$

The amplitudes k^{1-}, k^{1+} are given by expansions of the form

$$\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}D_y^\alpha\left((\partial_\xi^\alpha\mathsf{k}^{(\infty)})(t,x,\tilde{\nabla}_x\phi(t,s,x,y,\xi))\mathsf{e}(t,s,y,\xi)\right)_{y=x}$$

and satisfy, for every $p \ge 0$,

$$\begin{aligned} &\left\|\partial_x^\alpha\partial_\xi^\beta \mathsf{k}^{1\pm}(t,s,x,\xi)\right\| \\ &\quad\le \begin{cases} C_{\alpha\beta\varepsilon p}\left(\frac{\Lambda(t)}{\Lambda(s)}\right)^{K_+}\frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : t'_\xi \le s \le t, \\ C_{\alpha\beta\varepsilon p}\left(\frac{\Lambda(t)}{\Lambda(t_\xi)}\right)^{K_+}\frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : s \le t'_\xi \le t, \\ C_{\alpha\beta\varepsilon}t_\xi^{-1}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : s \le t \le t'_\xi. \end{cases} \end{aligned} \tag{3.14}$$

On the other hand,

$$\left|\partial_x^\alpha\partial_\xi^\beta e^{i\phi^\pm(t,s,x,\xi)-ix\xi}\right| \le \begin{cases} C_{\alpha\beta}\Lambda(t)^{|\alpha|+|\beta|}\langle\xi\rangle^{|\alpha|} & : t'_\xi \le t, \quad s \le t, \\ C_\beta\langle\xi\rangle^{-|\beta|} & : s \le t \le t_\xi. \end{cases}$$

Since (3.14) holds for every $p \geq 0$, the operator $\mathsf{K}^{(\infty)}(t)\mathsf{E}_1(t,s)$ is a pseudodifferential operator. The symbol of that operator (and its derivatives) can be bounded by the right-hand side of (3.14), for every $p \geq 0$. In a similar way we can consider the operator $\mathsf{E}_1(s,t)(\mathsf{K}^{(\infty)}(t)\mathsf{E}_1(t,s))$ and obtain (3.13). $\square$

REMARK 3.7. We can write (3.13) in the form

$$\left\| \partial_x^\alpha \partial_\xi^\beta \mathsf{k}^{(\infty)}(t,s,x,\xi) \right\| \leq C_{\alpha\beta\varepsilon p} g_p(t,x,\xi) \langle \xi \rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \quad \forall\, (t,s,x,\xi)$$

with

$$g_p(t,x,\xi) := \begin{cases} \left(\frac{\Lambda(t)}{\Lambda(t_\xi)}\right)^{2K_+} \frac{\lambda(t)}{\Lambda(t)} (\Lambda(t)\langle\xi\rangle)^{-p} & : t'_\xi \leq t, \\ t_\xi^{-1} & : t \leq t'_\xi. \end{cases}$$

By direct computation, $\int_0^T g_p(t,x,\xi)\,dt \leq C$ if $2K_+ - p < 0$.

Now we are in a position to describe Q_1.

Proposition 3.8. *There is a pseudodifferential operator* $\mathsf{Q}_1(t,s)$ *that solves* (3.12) *and whose matrix symbol belongs to* $L^\infty([0,T]^2, S^0_{1-\varepsilon,\varepsilon})$.

Proof. We look for q_1 in the form $\mathsf{q}_1 \sim \sum_{j=0}^\infty \mathsf{q}_{1j} \mod L^\infty([0,T]^2, S^{-\infty})$ where

$$D_t \mathsf{q}_{1m}(t,s,x,\xi) = \sum_{|\alpha|=0}^{m} \frac{1}{\alpha!} \left(D_\xi^\alpha \mathsf{k}^{(\infty)}(t,s,x,\xi)\right) \left(\partial_x^\alpha \mathsf{q}_{1(m-|\alpha|)}(t,s,x,\xi)\right)$$

with the initial conditions $\mathsf{q}_{10}(s,s) = I$ and $\mathsf{q}_{1m}(s,s) = 0$ for $m \geq 1$. We write this matrix ODE in the form

$$\partial_t \mathsf{q}_{1m}(t,s) = i\mathsf{k}^{(\infty)}(t,s)\mathsf{q}_{1m}(t,s) + \mathsf{r}_m(t,s)$$

and introduce the notations

$$b_0(t,s) := I, \quad b_{\nu+1}(t,s) := \int_s^t i\mathsf{k}(\tau,s) b_\nu(\tau,s)\,d\tau,$$
$$d_{0m}(t,s) := \int_s^t \mathsf{r}_m(\tau,s)\,d\tau, \quad d_{(\nu+1)m}(t,s) := \int_s^t i\mathsf{k}(\tau,s) d_{\nu m}(\tau,s)\,d\tau.$$

Then the representations

$$\mathsf{q}_{10}(t,s) = \sum_{\nu=0}^\infty b_\nu(t,s), \quad \mathsf{q}_{1m}(t,s) = \sum_{\nu=0}^\infty d_{\nu m}(t,s) \quad (m \geq 1)$$

hold. By means of induction, we show that

$$\left\|\partial_x^\alpha \partial_\xi^\beta b_\nu(t,s,x,\xi)\right\| \le C_{\alpha\beta\varepsilon} \langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \frac{1}{\nu!} \left(\int_s^t g_p(\tau,x,\xi)\,d\tau\right)^\nu$$

for all (t,s,x,ξ) with $t \ge s$ and all ν. This is true for $\nu = 0$. Then we have

$$\begin{aligned} &\left\|\partial_x^\alpha \partial_\xi^\beta b_{\nu+1}(t,s,x,\xi)\right\| \\ &\quad \le C_{\alpha\beta\varepsilon} \langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \frac{1}{\nu!} \int_s^t g_p(\tau,x,\xi) \left(\int_s^\tau g_p(\sigma,x,\xi)\,d\sigma\right)^\nu d\tau \\ &\quad = C_{\alpha\beta\varepsilon} \langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \frac{1}{(\nu+1)!} \left(\int_s^t g_p(\tau,x,\xi)\,d\tau\right)^{\nu+1}. \end{aligned}$$

Now we prove that

$$\left\|\partial_x^\alpha \partial_\xi^\beta \mathsf{q}_{1m}(t,s,x,\xi)\right\| \le C_{\alpha\beta\varepsilon m} \langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|-(1-2\varepsilon)m} \quad \forall\,(t,s,x,\xi). \tag{3.15}$$

Due to $\int_0^T g_p(t,x,\xi)\,dt \le C$ for large p, this holds for $m = 0$. Assuming that (3.15) is true for $m-1$, we show (3.15) for m. Clearly,

$$\left\|\partial_x^\alpha \partial_\xi^\beta \mathsf{r}_m(t,s,x,\xi)\right\| \le C_{\alpha\beta\varepsilon m} \langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} g_p(t,x,\xi).$$

Then it follows immediately that

$$\left\|\partial_x^\alpha \partial_\xi^\beta d_{0m}(t,s,x,\xi)\right\| \le C_{\alpha\beta\varepsilon m} \langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \int_s^t g_p(\tau,x,\xi)\,d\tau.$$

Similarly to the estimate of b_ν, we can show that

$$\left\|\partial_x^\alpha \partial_\xi^\beta d_{\nu m}(t,s,x,\xi)\right\| \le \frac{C_{\alpha\beta\varepsilon m} \langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|-(1-2\varepsilon)m}}{(\nu+1)!} \left(\int_s^t g_p(\tau,x,\xi)\,d\tau\right)^{\nu+1},$$

which implies (3.15).

So far, we have constructed $\mathsf{q}_{1m} \in L^\infty([0,T]^2, S^{-(1-2\varepsilon)m}_{1-\varepsilon,\varepsilon})$. Therefore, we find a symbol $\tilde{\mathsf{q}}_1 \sim \sum_{m=0}^\infty \mathsf{q}_{1m} \mod L^\infty([0,T]^2, S^{-\infty})$ which satisfies

$$D_t \tilde{\mathsf{q}}_1(t,s) - \mathsf{k}^{(\infty)}(t,s) \circ \tilde{\mathsf{q}}_1(t,s) = \mathsf{r}_\infty(t,s) \in L^\infty([0,T]^2, S^{-\infty}).$$

The solution Q_1 can be constructed from the approximative solution $\tilde{\mathsf{Q}}_1$ in a standard way, compare the proof of Proposition 3.4. □

4. The linear Cauchy problem

Let us consider the solution w to

$$Lw = f(t,x), \quad w(0,x) = u_0(x), \quad w_t(0,x) = u_1(x). \tag{4.1}$$

We define the weight symbol $\vartheta(t,x,\xi)$ by

$$\vartheta(t,x,\xi) = \exp\left(\int_t^{T_0} \frac{\lambda(\tau)}{\Lambda(\tau)} \chi_N^-(\tau,\xi)\beta(x)d\tau\right), \tag{4.2}$$

$$\beta(x) := \frac{l_*}{2(l_*+1)} \max_{\mp,\xi} \left| \left(1 \mp \frac{b+c}{\sqrt{c^2+a}}\right)(0,x,\xi)\right|, \tag{4.3}$$

compare (3.7). The number $\beta(x)$ tells us the amount of Sobolev regularity that was lost at the point x when we passed from $W(0,x)$ to $W(t,x)$ $(t > 0)$, where $W(t,x) = (G(t,D_x)w(t,x), D_t w(t,x))$, see (3.1). By computation,

$$\beta(x) = \frac{l_*}{2(l_*+1)} \left(1 + \left(l(x)^T (M(x))^{-1} l(x)\right)^{1/2}\right),$$

where $l(x)$ $(M(x))$ is the uniquely determined vector (symmetric matrix, respectively) such that

$$l(x)^T \xi = (b(0,x,\xi) + c(0,x,\xi))|\xi|, \quad \xi^T M(x)\xi = (c(0,x,\xi)^2 + a(0,x,\xi))|\xi|^2.$$

Observe that $M(x)$ is positively definite, uniformly in x, due to (1.9).

ASSUMPTION A. The vector $(c_j(0,x) - b_j(0,x))_{j=1}^n$ either vanishes identically on $\mathbb{R}_x^n$, or it never vanishes.

Under this assumption, $\beta(x) \in C^\infty(\mathbb{R}^n)$. Now we choose the Sobolev space of variable order $B_{\vartheta,M,T}$ and space $C_{\vartheta,M}$ of traces of functions of $B_{\vartheta,M,T}$ at $t = 0$,

$$\|W\|_{M,T} := \sup_{[0,T]}(\|\Theta_M(t,x,D_x)W(t,x)\|_{L^2} + \|W(t,x)\|_{L^2}),$$
$$\|W(0)\|_{C_{\vartheta,M}} := \|\Theta_M(0,x,D_x)W(0,x)\|_{L^2} + \|W(0,x)\|_{L^2}.$$

The main result of this section is the following *a priori* estimate:

Theorem 4.1. *Suppose* Assumption A, $u_0 \in C_{\vartheta,M+1/(l_*+1)}$ *and* $u_1 \in C_{\vartheta,M}$ *with* $M \geq 0$. *Then there are a time interval* $[0,T]$ *and a constant* C_{apr} *with the property that a solution* $w \in B_{\vartheta,M+1/(l_*+1),T}$ *to* (4.1) *exists and satisfies*

$$\|W\|_{M,T} \leq C_{apr}(\|W(0)\|_{C_{\vartheta,M}} + T\,\|f\|_{M,T}),$$

where $W(t,x) = (G(t,D_x)w(t,x), D_t w(t,x))^T$, *see* (3.1).

The proof is split into several lemmas.

Lemma 4.2. *If* $\mathscr{H}^{\mp}_{s,t}(x,\xi)$ *denotes the Hamilton flow of* Definition 3.2 *and* $0 \le s$, $\sigma \le t \le T$, *then*

$$\frac{\vartheta(t,\mathscr{H}^{\mp}_{s,\sigma}(x,\xi))}{\vartheta(t,x,\xi)} \in L^\infty([0,T]^3, S^0_{1-\varepsilon,\varepsilon}).$$

Proof. Due to Hadamard's formula, we can write

$$\begin{aligned}&\frac{\lambda(\tau)}{\Lambda(\tau)}(\chi_N^-\beta)(\tau,\mathscr{H}^{\mp}_{s,\sigma}(x,\xi))\\ &\quad = \frac{\lambda(\tau)}{\Lambda(\tau)}(\chi_N^-\beta)(\tau,x,\xi) + \frac{\lambda(\tau)}{\Lambda(\tau)}r(\tau,s,\sigma,x,\xi)(s-\sigma),\end{aligned}$$

with some remainder $r \in L^\infty([0,T]^3, S^0_{1,0})$. Our assumptions and $t \le \tau$ show that $\lambda(\tau)|s-\sigma|/\Lambda(\tau) \le C$, which yields the assertion. □

The same argument gives the following improvement of Proposition 3.5.

Lemma 4.3. *The symbol* $\operatorname{diag}(\mathsf{q}_0^-, \mathsf{q}_0^+)$ *of the operator* $\mathsf{Q}_0(t,s)$ *has the form*

$$\begin{aligned}&\mathsf{q}_0^{\mp}(t,s,x,\xi)\\ &\quad = \exp\left(\int_s^t \frac{\lambda'(\tau)}{2\lambda(\tau)}\chi_N^-(\tau,\xi)\left(1 \mp \frac{b+c}{\sqrt{c^2+a}}\right)(0,x,\xi)\,d\tau\right) \circ \tilde{\mathsf{q}}_0^{\mp}(t,s,x,\xi),\end{aligned}$$

where $\tilde{\mathsf{q}}_0^{\mp} \in L^\infty([0,T]^2, S^0_{1-\varepsilon,\varepsilon})$.

The next lemma is a variant of Egorov's theorem.

Lemma 4.4. *There are operators* $\mathsf{R}(t,s) \in L^\infty([0,T]^2, \Psi^0_{1-\varepsilon,\varepsilon})$ *and* $\mathsf{R}_\infty(t,s) \in L^\infty([0,T]^2, \Psi^{-\infty})$ *with the property that*

$$\mathsf{E}_0(s,t)\Theta_M(t)\mathsf{E}_0(t,s) = \mathsf{R}(t,s)\Theta_M(t) + \mathsf{R}_\infty(t,s), \quad 0 \le s \le t \le T.$$

Proof. In the sequel, we consider t as fixed and let s be running in $[0,t]$. We write $\tilde\Theta(t,s) := \mathsf{E}_0(s,t)\Theta_M(t)\mathsf{E}_0(t,s)$ and obtain

$$D_s\tilde\Theta(t,s) = [\mathsf{D}(s),\tilde\Theta(t,s)] = \mathsf{D}(s)\tilde\Theta(t,s) - \tilde\Theta(t,s)\mathsf{D}(s).$$

We look for $\tilde{\Theta}$ having the form

$$(\tilde{\Theta}(t,s)W)(x) = \int e^{ix\xi}\tilde{\vartheta}(t,s,x,\xi)\hat{W}(\xi)\,d\xi, \quad \tilde{\vartheta} \sim \sum_{m=0}^{\infty}\tilde{\vartheta}_m,$$

where $\tilde{\vartheta}_m = \mathrm{diag}(\tilde{\vartheta}_m^-, \tilde{\vartheta}_m^+)$ are diagonal symbols satisfying

$$\begin{aligned}
&\partial_s\tilde{\vartheta}_m(t,s,x,\xi)\\
&\quad = \sum_{|\alpha|=1}\left(\partial_\xi^\alpha \mathsf{D}(s,x,\xi)\right)\left(\partial_x^\alpha\tilde{\vartheta}_m(t,s,x,\xi)\right) - \left(\partial_x^\alpha \mathsf{D}(s,x,\xi)\right)\left(\partial_\xi^\alpha\tilde{\vartheta}_m(t,s,x,\xi)\right)\\
&\qquad + \sum_{l=0}^{m-1}\sum_{|\alpha|=m+1-l}\frac{i}{\alpha!}\Big(\left(D_\xi^\alpha \mathsf{D}(s,x,\xi)\right)\left(\partial_x^\alpha\tilde{\vartheta}_l(t,s,x,\xi)\right)\\
&\qquad\qquad - \left(\partial_x^\alpha \mathsf{D}(s,x,\xi)\right)\left(D_\xi^\alpha\tilde{\vartheta}_l(t,s,x,\xi)\right)\Big)
\end{aligned}$$

with the initial conditions $\tilde{\vartheta}_0^\mp(t,t,x,\xi) = \vartheta_M(t,x,\xi)$, $\tilde{\vartheta}_m^\mp(t,t,x,\xi) = 0$ $(m \geq 1)$. This first order PDE can be written as

$$\partial_s\tilde{\vartheta}_m^\mp(t,s,x,\xi) = \mathscr{H}^\mp\tilde{\vartheta}_m^\mp(t,s,x,\xi) + f_m^\mp(t,s,x,\xi), \quad f_0^\mp \equiv 0,$$

where $\mathscr{H}^\mp = \sum_{|\alpha|=1}(\partial_\xi^\alpha\tau^\mp)\partial_x^\alpha - (\partial_x^\alpha\tau^\mp)\partial_\xi^\alpha$ is the Hamilton vector field. Then the solutions $\tilde{\vartheta}_m^\mp$ are given by

$$\begin{aligned}
\tilde{\vartheta}_0^\mp(t,s,x,\xi) &= \vartheta_M(t, \mathscr{H}_{s,t}^\mp(x,\xi)),\\
\tilde{\vartheta}_m^\mp(t,s,x,\xi) &= \int_t^s f_m^\mp(t,\sigma,\mathscr{H}_{s,\sigma}^\mp(x,\xi))\,d\sigma, \quad 0 \leq s \leq t \leq T, \quad m \geq 1.
\end{aligned}$$

From Lemma 4.2, we deduce that $\tilde{\vartheta}_0^\mp(t,s,x,\xi) = r_0^\mp(t,s,x,\xi)\vartheta_M(t,x,\xi)$ with $r_0^\mp \in L^\infty([0,T]^2, S^0_{1-\varepsilon,\varepsilon})$. By $\vartheta_M(t,.,.) \in \mathcal{K}_{1-\varepsilon,\varepsilon}$, it can be concluded that $f_1^\mp(t,s,x,\xi) = \tilde{f}_1^\mp(t,s,x,\xi)\vartheta_M(t,x,\xi)$, with $\tilde{f}_1^\mp \in L^\infty([0,T]^2, S^{-(1-2\varepsilon)}_{1-\varepsilon,\varepsilon})$. Applying Lemma 4.2 again shows $\tilde{\vartheta}_1^\mp(t,s,x,\xi) = r_1^\mp(t,s,x,\xi)\vartheta_M(t,x,\xi)$, where $r_1^\mp \in L^\infty([0,T]^2, S^{-(1-2\varepsilon)}_{1-\varepsilon,\varepsilon})$. Following this procedure we get

$$\tilde{\vartheta}_m^\mp(t,s,x,\xi) = r_m^\mp(t,s,x,\xi)\vartheta_M(t,x,\xi), \quad r_m^\mp \in L^\infty([0,T]^2, S^{-(1-2\varepsilon)m}_{1-\varepsilon,\varepsilon}).$$

We define $\vartheta^{*,\mp} \sim (\sum_{m=0}^\infty r_m^\mp)\vartheta_M$, which gives us an approximative solution,

$$D_s\Theta^*(t,s) - [\mathsf{D}(s), \Theta^*(t,s)] = \mathsf{R}_{0,\infty}(t,s) \in L^\infty([0,T]^2, \Psi^{-\infty}).$$

It remains to show that $\tilde{\Theta} - \Theta^*$ is a smoothing operator. This is equivalent to prove that $W(t,s) := \mathsf{E}_0(s,t)\Theta_M(t) - \Theta^*(t,s)\mathsf{E}_0(s,t)$ smooths. We have

$$(D_s - \mathsf{D}(s))W(t,s) = -\mathsf{R}_{0,\infty}(t,s)\mathsf{E}_0(s,t)$$

and $W(t,t) = 0$. Then Duhamel's principle yields

$$W(t,s) = -\int_t^s \mathsf{E}_0(s,\tau)\mathsf{R}_{0,\infty}(t,\tau)\mathsf{E}_0(\tau,t)\,d\tau \in \Psi^{-\infty}.$$

This completes the proof. □

Proof of Theorem 4.1. The vector W can be written as

$$\begin{aligned} W(t,x) =& \mathsf{E}(t,0)W(0,x) + \int_0^t \mathsf{E}(t,s)F(s,x)\,ds \\ =& \mathsf{M}(t)\mathsf{N}(t)\mathsf{E}_0(t,0)\mathsf{Q}_0(t,0)\mathsf{Q}_1(t,0)\mathsf{N}^\sharp(0)\mathsf{M}^\sharp(0)W(0,x) \\ &+ \int_0^t \mathsf{M}(t)\mathsf{N}(t)\mathsf{E}_0(t,s)\mathsf{Q}_0(t,s)\mathsf{Q}_1(t,s)\mathsf{N}^\sharp(s)\mathsf{M}^\sharp(s)F(s,x)\,ds \\ &+ \mathsf{R}_\infty(t,0)W(0,x) + \int_0^t \mathsf{R}_\infty(t,s)F(s,x)\,ds. \end{aligned}$$

Lemma 2.5 shows that there are $\tilde{\mathsf{M}}(t), \tilde{\mathsf{N}}(t) \in L^\infty([0,T], \Psi^0_{1-\varepsilon,\varepsilon})$, and $\tilde{\mathsf{Q}}_1(t,s) \in L^\infty([0,T]^2, \Psi^0_{1-\varepsilon,\varepsilon})$, such that modulo regularizing operators we have

$$\Theta_M(t)\mathsf{M}(t)\mathsf{N}(t) = \tilde{\mathsf{M}}(t)\tilde{\mathsf{N}}(t)\Theta_M(t), \quad \Theta_M(s)\mathsf{Q}_1(t,s) = \tilde{\mathsf{Q}}_1(t,s)\Theta_M(s).$$

Then we have, modulo smoothing operators,

$$\Theta_M(t)\mathsf{M}(t)\mathsf{N}(t)\mathsf{E}_0(t,s)\mathsf{Q}_0(t,s) = \tilde{\mathsf{M}}(t)\tilde{\mathsf{N}}(t)\mathsf{E}_0(t,s)\mathsf{R}(t,s)\Theta_M(t)\mathsf{Q}_0(t,s),$$

see Lemma 4.4. From Lemma 4.3 and the choice of ϑ, it follows that

$$\Theta_M(t)\mathsf{Q}_0(t,s) = \mathsf{R}_0(t,s)\Theta_M(s) + \mathsf{R}_\infty(t,s),$$

with some $\mathsf{R}_0 \in L^\infty([0,T]^2, \Psi^0_{1-\varepsilon,\varepsilon})$. As a summary, we have

$$\begin{aligned} &\Theta_M(t)\mathsf{M}(t)\mathsf{N}(t)\mathsf{E}_0(t,s)\mathsf{Q}_0(t,s)\mathsf{Q}_1(t,s)\mathsf{N}^\sharp(s)\mathsf{M}^\sharp(s) \\ &\quad = \mathsf{R}(t,s)\Theta_M(s) + \mathsf{R}_\infty(t,s), \quad 0 \le s \le t \le T, \end{aligned}$$

with some $\mathsf{R}(t,s) \in L^\infty([0,T]^2, \mathcal{L}(L^2))$. Then it follows that

$$\begin{aligned} \|\Theta_M(t)W(t,x)\|^2 \le& C\,\|\mathsf{R}(t,0)\Theta_M(0)W(0,x)\|^2 + C\,\|\mathsf{R}_\infty(t,0)W(0,x)\|^2 \\ &+ Ct\int_0^t \|\mathsf{R}(t,s)\Theta_M(s)F(s,x)\|^2\,ds + Ct\int_0^t \|\mathsf{R}_\infty(t,s)F(s,x)\|^2\,ds. \end{aligned}$$

Integration over $\mathbb{R}^n_x$ gives the desired inequality.

The weight symbol $g(t,\xi)$ of (3.1) satisfies $g(t,\xi) \ge C\langle\xi\rangle^{1/(l_*+1)}$, $C > 0$. Then we have $\|w\|_{M+1/(l_*+1),T} \le C\,\|Gw\|_{M,T}$; hence $w \in B_{\vartheta,M+1/(l_*+1),T}$. □

5. The algebra property

One variant of the extension of the result of Theorem 4.1 to semilinear Cauchy problems relies on the algebra property of the space $B_{\vartheta,M,T}$. This idea was used in [7], where Cauchy problems similar to (1.6) and (1.7) were studied. In this section, the algebra property of the spaces $B_{\vartheta,M,T}$ is proved.

Theorem 5.1. *Assume $M > n/2 + 1$. Then there is a constant C_0 such that*

$$\|w_1 w_2\|_{M,T} \leq C_0 \|w_1\|_{M,T} \|w_2\|_{M,T}$$

for all w_1, $w_2 \in B_{\vartheta,M,T}$.

Corollary 5.2. *Let $f(t,x,u)$ satisfy* (1.10), (1.11) *and suppose $M > n/2 + 1$. Then there is, for each $K > 0$, a constant $C_1(K)$ with the property that*

$$\|f(t,x,u) - f(t,x,v)\|_{M,T} \leq C_1(K) \|u - v\|_{M,T}$$

provided that $u, v \in B_{\vartheta,M,T}$ and $\|u\|_{M,T}$, $\|v\|_{M,T} \leq K$.

The proof is split into several parts. In Lemma 5.3 and Corollary 5.4, we replace the operator Θ_M by a new operator whose Schwartz Kernel has support close to the diagonal $\{x = y\}$ of $\mathbb{R}^n_x \times \mathbb{R}^n_y$. Then this new localized operator is locally decomposed into a product of two operators; the first is an operator of small order, the second has a symbol independent of x. Exploiting an estimate given in Proposition 5.5, the desired inequality is proved locally in Lemma 5.6. Finally, all these local estimates are glued together in Lemma 5.7.

Let $\{B_\varrho(x_k^0)\}_{k=0}^\infty$ be a locally finite covering of $\mathbb{R}^n$ with balls of radius ϱ and center x_k^0 and let $\{\varphi_k(x)\}_{k=0}^\infty$ be its associated partition of unity. Fix some $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ for $|x| \leq 2$, $\psi(x) = 0$ for $|x| \geq 3$ and define $\psi_\varrho(x) = \psi(x/\varrho)$.

Lemma 5.3. *For $p \in S^m_{1,\varepsilon}$, define $P_\varrho = P_\varrho(x, D_x)$ by*

$$(P_\varrho w)(x) = \sum_{k=0}^\infty \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\xi} \varphi_k(x) p(x,\xi)(1-\psi_\varrho)(y - x_k^0) w(y)\, dy\, d\xi.$$

Then P_ϱ is a smoothing operator, $P_\varrho \in \Psi^{-\infty}$.

Proof. The Schwartz Kernel of P_ϱ,

$$\sum_{k=0}^\infty \int_{\mathbb{R}^n_\xi} e^{i(x-y)\xi} \varphi_k(x) p(x,\xi)(1-\psi_\varrho)(y - x_k^0)\, d\xi,$$

vanishes for $|x-y| \leq \varrho$. $\square$

Corollary 5.4. *The following norm is an equivalent norm for* $B_{\vartheta,M,T}$:

$$\|w\|^2_{M,T} \sim \sup_{[0,T]} \left(\sum_{k=0}^{\infty} \left\| \varphi_k(x)\Theta_M(t,x,D_x)(\psi_\varrho(x-x_k^0)w(t,x)) \right\|^2_{L^2} + \|w\|^2_{L^2} \right).$$

Proposition 5.5. *Suppose that* $p(t,\xi) \in L^\infty([0,T], S^m_{1,0})$ *satisfies*

(5.1) $\quad p(t,\xi) = p(t,|\xi|)$ *is monotonically increasing in* $|\xi|$,

(5.2) $\quad p(t,2\xi) \leq C_0 p(t,\xi)$,

(5.3) $\quad C_0 p(t,\xi) \geq \langle\xi\rangle^{n/2+1+\gamma}, \quad \gamma > 0$,

(5.4) $\quad |\nabla_\xi p(t,\xi)| \leq C_0 p(t,\xi)\langle\xi\rangle^{-1}$

for all $(t,\xi) \in [0,T] \times \mathbb{R}^n$ *and some* $C_0 > 0$. *If* $\delta \geq 0$, *then the estimate*

$$\|P(fg) - fPg - gPf\|_{H^\delta} \leq C_1 \|Pf\|_{H^{\delta-1/2}} \|Pg\|_{H^{\delta-1/2}}$$

holds for all f, g *with* $Pf,\ Pg \in H^{\delta-1/2}$; *and* $C_1 = C_1(C_0, n, \gamma, \delta)$.

Proof. If w is an arbitrary function of $L^2(\mathbb{R}^n_\xi)$, then

$$\begin{aligned}
&\int_{\mathbb{R}^n_\xi} \langle\xi\rangle^\delta (P(fg) - fPg - gPf)\widehat{\ }(\xi)w(\xi)\,d\xi \\
&\quad = \int_{\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_\eta} (\langle\xi-\eta\rangle^{\delta-1/2} p(\xi-\eta)\hat f(\xi-\eta))(\langle\eta\rangle^{\delta-1/2} p(\eta)\hat g(\eta))\times \\
&\qquad \times m(\xi,\eta)w(\xi)\,d\eta\,d\xi
\end{aligned}$$

holds, where we have neglected the variable t and

$$m(\xi,\eta) := \frac{|p(\xi) - p(\xi-\eta) - p(\eta)|\langle\xi\rangle^\delta\langle\eta\rangle^{1/2}\langle\xi-\eta\rangle^{1/2}}{p(\xi-\eta)p(\eta)\langle\eta\rangle^\delta\langle\xi-\eta\rangle^\delta}.$$

Assume that we had shown

$$\sup_{\xi,t} \int_{\mathbb{R}^n_\eta} m(\xi,\eta)^2 d\eta = C_1^2 < \infty. \tag{5.5}$$

Then the Cauchy-Schwarz inequality implies

$$\left| \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^\delta (P(fg) - fPg - gPf)\widehat{\ }(\xi)w(\xi)\,d\xi \right|$$

$$
\begin{aligned}
&\leq \left(\int_{\mathbb{R}^n_\eta} \int_{\mathbb{R}^n_\xi} |\langle \xi - \eta \rangle^{\delta - 1/2} p(\xi - \eta) \hat{f}(\xi - \eta) \langle \eta \rangle^{\delta - 1/2} p(\eta) \hat{g}(\eta)|^2 d\xi \, d\eta \right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_\eta} m(\xi, \eta)^2 |w(\xi)|^2 d\eta \, d\xi \right)^{1/2} \\
&\leq C_1 \|w\|_{L^2(\mathbb{R}^n_\xi)} \|Pf\|_{H^{\delta - 1/2}} \|Pg\|_{H^{\delta - 1/2}},
\end{aligned}
$$

which completes the proof. Therefore, it remains to show (5.5).

For each ξ, we split $\mathbb{R}^n_\eta$ into four parts:

$$
\begin{aligned}
&A = \{\eta : |\eta| \geq 2|\xi|\}, \quad B = \left\{\eta : |\eta| \leq 2|\xi|, \ |\xi - \eta| \leq \frac{|\eta|}{2}\right\}, \\
&C = \{\eta : |\eta| \leq 2|\xi|, \ |\xi - \eta| \geq 2|\eta|\}, \\
&D = \left\{\eta : |\eta| \leq 2|\xi|, \ \frac{|\eta|}{2} \leq |\xi - \eta| \leq 2|\eta|\right\}.
\end{aligned}
$$

If $\eta \in A$, then $|\xi - \eta| \geq |\eta|/2$; hence

$$
m(\xi, \eta) \leq C \frac{p(\eta) \langle \eta \rangle^{1/2} \langle \xi - \eta \rangle^{1/2}}{p(\xi - \eta) p(\eta)} \leq C \frac{\langle \xi - \eta \rangle}{p(\xi - \eta)} \leq \frac{C}{\langle \eta \rangle^{n/2 + \gamma}}.
$$

This implies $\int_A m(\xi, \eta)^2 d\eta \leq C$.

If $|\xi - \eta| \leq |\xi|/2$, then $|\xi|/2 \leq |\eta| \leq 3|\xi|/2$ and $|\xi - \eta| \leq |\eta|$. Consequently, (ξ, η) is the shortest side in the triangle with the corners $0, \xi, \eta$. Then we can find a positive number C with

$$
C \min(|\eta|, |\xi|) \leq |\alpha \eta + (1 - \alpha) \xi| \leq \max(|\eta|, |\xi|), \quad \forall \, 0 \leq \alpha \leq 1.
$$

From this and (5.4), we deduce that

$$
\begin{aligned}
|p(\xi) - p(\eta)| &\leq \max_{0 \leq \alpha \leq 1} |\nabla p(\alpha \xi + (1 - \alpha) \eta)| \cdot |\xi - \eta| \\
&\leq C \max_{0 \leq \alpha \leq 1} \frac{p(\alpha \xi + (1 - \alpha) \eta)}{\langle \alpha \xi + (1 - \alpha) \eta \rangle} \cdot \langle \xi - \eta \rangle,
\end{aligned}
$$

$$
|p(\xi) - p(\eta)| \leq C \frac{p(\xi)}{\langle \xi \rangle} \langle \xi - \eta \rangle \qquad \forall \, |\xi - \eta| \leq \frac{|\xi|}{2}. \tag{5.6}
$$

Now let $\eta \in B$. From (5.6), we get

$$
\begin{aligned}
|p(\xi) - p(\eta) - p(\xi - \eta)| &\leq C \frac{p(\eta)}{\langle \eta \rangle} \langle \xi - \eta \rangle + p(\xi - \eta), \\
m(\xi, \eta) &\leq C \frac{p(\eta) \langle \xi - \eta \rangle^{3/2} \langle \eta \rangle^{1/2}}{p(\xi - \eta) p(\eta) \langle \eta \rangle} + C \frac{p(\xi - \eta) \langle \eta \rangle^{1/2} \langle \xi - \eta \rangle^{1/2}}{p(\xi - \eta) p(\eta)}
\end{aligned}
$$

$$\leq C\frac{\langle\xi-\eta\rangle}{p(\xi-\eta)}+C\frac{\langle\eta\rangle}{p(\eta)}.$$

This implies $\int_B m(\xi,\eta)^2\,d\eta \leq C$.

The case $\eta\in C$ is treated similarly; we just change the roles of η and $\xi-\eta$.

Finally, if $\eta\in D$, then

$$m(\xi,\eta)\leq C\frac{p(\eta)\langle\eta\rangle^{1/2}\langle\xi-\eta\rangle^{1/2}}{p(\xi-\eta)p(\eta)}\leq C\frac{\langle\xi-\eta\rangle}{p(\xi-\eta)}\leq\frac{C}{\langle\eta\rangle^{n/2+\gamma}}.$$

The proof is complete. ☐

We define two additional cut-off functions,

$$\zeta^{(1)}(x)=\begin{cases}1 & :|x|\leq 3,\\ 0 & :|x|\geq 4,\end{cases}\qquad \zeta^{(2)}(x)=\begin{cases}1 & :|x|\leq 5,\\ 0 & :|x|\geq 6,\end{cases}$$

and set $\zeta^{(j)}_\varrho(x):=\zeta^{(j)}(x/\varrho)$, $j=1,2$. Then we introduce the notations $w_{1,k}(x)=\zeta^{(1)}_\varrho(x-x^0_k)w_1(x)$, $w_{2,k}(x)=\psi_\varrho(x-x^0_k)w_2(x)$, and it follows that $\psi_\varrho(x-x^0_k)w_1(x)w_2(x)=w_{1,k}(x)w_{2,k}(x)$. We define the symbol

$$\vartheta_{(k)}(t,x,\xi)=\exp\left(\int_t^T\frac{\lambda(\tau)}{\Lambda(\tau)}\chi^-_N(\tau,\xi)(\beta(x)-\beta(x^0_k))\,d\tau\right)$$

and conclude that $\Theta_M(t,x,D_x)=\Theta_{(k)}(t,x,D_x)\Theta_M(t,x^0_k,D_x)$.

Lemma 5.6. *Let δ be a positive number with*

$$\sup_k\sup_{|x-x^0_k|\leq 8\varrho}\left|\beta(x)-\beta(x^0_k)\right|\leq\delta\leq\frac{1}{4}. \tag{5.7}$$

Then there is some constant $C_{\delta,\varrho}$, independent of t, k, and w_j, such that

$$\begin{aligned}&\left\|\varphi_k(x)\Theta_M(t,x,D_x)(\psi_\varrho(x-x^0_k)w_1(x)w_2(x))\right\|_{L^2}\\ &\quad\leq C_{\delta,\varrho}(\left\|\zeta^{(2)}_\varrho(x-x^0_k)\Theta_M(t,x,D_x)w_{1,k}(x)\right\|_{L^2}+\|w_{1,k}(x)\|_{L^2})\times\\ &\qquad\times(\left\|\zeta^{(2)}_\varrho(x-x^0_k)\Theta_M(t,x,D_x)w_{2,k}(x)\right\|_{L^2}+\|w_{2,k}(x)\|_{L^2}).\end{aligned}$$

Proof. We have the decomposition

$$\begin{aligned}\varphi_k\Theta_M(t,x,D_x)(w_{1,k}w_{2,k})&=I_{k,1}+I_{k,2}+I_{k,3}\\ &=\varphi_k\Theta_{(k)}(\Theta_M(t,x^0_k,D_x)(w_{1,k}w_{2,k})-w_{1,k}\Theta_M(t,x^0_k,D_x)w_{2,k}\\ &\qquad-w_{2,k}\Theta_M(t,x^0_k,D_x)w_{1,k})\end{aligned}$$

$$+ \varphi_k \Theta_{(k)}(w_{1,k} \Theta_M(t, x_k^0, D_x) w_{2,k}) + \varphi_k \Theta_{(k)}(w_{2,k} \Theta_M(t, x_k^0, D_x) w_{1,k}).$$

By the choice of δ, we deduce that $\vartheta_{(k)}(t, x, \xi) \leq C\langle\xi\rangle^\delta$, as long as $|x - x_k^0| \leq 8\varrho$. Then it follows that $\varphi_k \vartheta_{(k)} \in L^\infty([0, T], S^\delta_{1,\varepsilon})$. Proposition 5.5 gives

$$\|I_{k,1}\|_{L^2} \leq C \left\|\Theta_M(t, x_k^0, D_x) w_{1,k}\right\|_{H^{\delta-1/2}} \left\|\Theta_M(t, x_k^0, D_x) w_{2,k}\right\|_{H^{\delta-1/2}} .$$

Now we show that

$$\begin{aligned} &\left\|\Theta_M(t, x_k^0, D_x) w_{j,k}\right\|_{H^{\delta-1/2}} \\ &\quad \leq C \left\|\zeta_\varrho^{(2)}(x - x_k^0) \Theta_M(t, x, D_x) w_{j,k}\right\|_{L^2} + C \|w_{j,k}\|_{L^2} . \end{aligned} \tag{5.8}$$

According to Proposition 2.4, the operator $\Theta_{(k)}$ has a parametrix $\Theta^\sharp_{(k)}$. Then we can write

$$\begin{aligned} \Theta_M(t, x_k^0, D_x) w_{j,k} &= (\Theta^\sharp_{(k)} \Theta_M)(t, x, D_x) w_{j,k} + R_\infty w_{j,k} \\ &= \Theta^\sharp_{(k)}((1 - \zeta_\varrho^{(2)}(x - x_k^0) + \zeta_\varrho^{(2)}(x - x_k^0)) \Theta_M w_{j,k}) + R_\infty w_{j,k}. \end{aligned}$$

Since the symbol of the operator $\Theta^\sharp_{(k)}(1 - \zeta_\varrho^{(2)}(x - x_k^0))\Theta_M$ vanishes on the support of $w_{j,k}$ (modulo smoothing operators), we see that

$$\left\|\Theta^\sharp_{(k)}((1 - \zeta_\varrho^{(2)}(x - x_k^0)) \Theta_M w_{j,k})\right\|_{H^{\delta-1/2}} \leq C \|w_{j,k}\|_{L^2} . \tag{5.9}$$

The order of the operator $\Theta^\sharp_{(k)}$ is at most δ in a neighbourhood of the support of $\zeta_\varrho^{(2)}(\,\cdot\, - x_k^0)$. Therefore,

$$\left\|\Theta^\sharp_{(k)}(\zeta_\varrho^{(2)}(x - x_k^0) \Theta_M w_{j,k})\right\|_{H^{\delta-1/2}} \leq C \left\|\zeta_\varrho^{(2)}(x - x_k^0) \Theta_M w_{j,k}\right\|_{L^2} .$$

This proves (5.8) and yields the estimate of $I_{k,1}$.

For each operator $P \in \Psi^\delta_{1,\varepsilon}$ ($\delta < 1$), there is a constant C such that for all Lipschitz continuous functions f and all $g \in H^{\delta-1}$ the estimate

$$\|[P, f]g\|_{L^2} \leq C \|\nabla f\|_{L^\infty} \|g\|_{H^{\delta-1}} \tag{5.10}$$

holds. This is a special case of Theorem 5.1 in [13]. Using (5.10) and

$$I_{k,2} = w_{1,k} \varphi_k \Theta_M(t, x, D_x) w_{2,k} + [\varphi_k \Theta_{(k)}, w_{1,k}] \Theta_M(t, x_k^0, D_x) w_{2,k},$$

we deduce that

$$\|I_{k,2}\|_{L^2} \leq C \|w_{1,k}\|_{C^1} (\|\varphi_k \Theta_M w_{2,k}\|_{L^2} + \left\|\Theta_M(t, x_k^0, D_x) w_{2,k}\right\|_{H^{\delta-1}}).$$

From $\langle\xi\rangle^M \le \vartheta_M(t,x,\xi)$ and $M > n/2+1$, we obtain

$$\|w_{1,k}\|_{C^1} \le C\left\|(1-\zeta_\varrho^{(2)}(x-x_k^0)+\zeta_\varrho^{(2)}(x-x_k^0))\Theta_M(t,x,D_x)w_{1,k}\right\|_{L^2} + C\|w_{1,k}\|_{L^2}.$$

The application of the reasoning that led to (5.9) implies

$$\|w_{1,k}\|_{C^1} \le C\left\|\zeta_\varrho^{(2)}(x-x_k^0)\Theta_M(t,x,D_x)w_{1,k}\right\|_{L^2} + C\|w_{1,k}\|_{L^2}.$$

The term $I_{k,3}$ can be treated similarly. □

Lemma 5.7. *Under the conditions of* Lemma 5.6, *we have*

$$\sum_{k=0}^{\infty}\left\|\varphi_k(x)\Theta_M(t,x,D_x)(\psi_\varrho(x-x_k^0)w_1(x)w_2(x))\right\|_{L^2}^2 \le C_{\delta,\varrho}\|w_1\|_{M,T}^2\|w_2\|_{M,T}^2 \quad \forall\, w_1,w_2\in B_{\vartheta,M,T}.$$

Proof. It suffices to show

$$\sum_{k=0}^{\infty}\left\|\zeta_\varrho^{(2)}(x-x_k^0)\Theta_M(t,x,D_x)w_{j,k}\right\|_{L^2}^2 \le C\|w_j\|_{M,T}^2. \tag{5.11}$$

Applying Lemma 2.5 twice, we find an operator $R_{1,k}\in L^\infty([0,T],\Psi^0_{1-\varepsilon,\varepsilon})$ with

$$\begin{aligned}\Theta_M(t,x,D_x)\circ\zeta_\varrho^{(1)}(x-x_k^0) &= R_{1,k}(t,x,D_x)\Theta_M(t,x,D_x)+R_\infty\\ &= R_{1,k}(1-\zeta_\varrho^{(3)}(x-x_k^0)+\zeta_\varrho^{(3)}(x-x_k^0))\Theta_M(t,x,D_x)+R_\infty,\end{aligned}$$

where $\zeta_\varrho^{(3)}(x)=\zeta^{(3)}(x/\varrho)$ and $\zeta^{(3)}(x)=1$ for $|x|\le 7$, $\zeta^{(3)}(x)=0$ for $|x|\ge 8$. Exploiting the idea behind Lemma 5.3 and Corollary 5.4, we get

$$\sum_{k=0}^{\infty}\left\|\zeta_\varrho^{(2)}(x-x_k^0)R_{1,k}((1-\zeta_\varrho^{(3)}(x-x_k^0))\Theta_M w_1)\right\|_{L^2}^2 \le C_\varrho\|w_1\|_{L^2}^2.$$

On the other hand, $\zeta_\varrho^{(2)}(x-x_k^0)R_{1,k}$ is an operator of order zero, uniformly bounded with respect to k; hence

$$\begin{aligned}&\sum_{k=0}^{\infty}\left\|\zeta_\varrho^{(2)}(x-x_k^0)R_{1,k}(\zeta_\varrho^{(3)}(x-x_k^0)\Theta_M w_1)\right\|_{L^2}^2\\ &\quad\le C\sum_{k=0}^{\infty}\left\|\zeta_\varrho^{(3)}(x-x_k^0)\Theta_M w_1\right\|_{L^2}^2 \le C\|w_1\|_{M,T}^2.\end{aligned}$$

This gives (5.11) for $j = 1$. The case $j = 2$ runs similarly. □

Proof of Theorem 5.1. We apply Corollary 5.4 and Lemma 5.7. □

6. The semilinear Cauchy problem

Theorem 4.1 gives us the existence of a solution v to the Cauchy problem (1.7) in some space $B_{\vartheta,M+1(l_*+1),T}$ provided $u_0 \in C_{\vartheta,M+1/(l_*+1)}$, $u_1 \in C_{\vartheta,M}$. The main goal of this section is to show that, for small time T, a solution u to (1.6) exists and belongs to the same space as v. Moreover, we prove that the difference $u - v$ has higher regularity than u and v, see Theorem 6.2. This implies that the strongest singularities of u and v are the same.

Theorem 6.1. *Suppose* Assumption A *and* $u_0 \in C_{\vartheta,M+1/(l_*+1)}$, $u_1 \in C_{\vartheta,M}$, *and* $M > n/2 + 1$. *Then there is a* T, $0 < T \leq T_0$, *such that a solution* $u \in B_{\vartheta,M+1/(l_*+1),T}$ *to* (1.6) *exists with* $U \in B_{\vartheta,M,T}$, *where* $U(t,x) = (G(t,D_x)u(t,x), D_t u(t,x))^T$.

Proof. We consider the mapping $\mathcal{A} : w \mapsto u$ defined by

$$Lu = f(t,x,w), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),$$

and show that it has a fixed point in $B_{\vartheta,M+1/(l_*+1),T}$ for small T, $0 < T \leq T_0$. Fix a constant C_2 such that

$$\|w(t,x)\|_{M,T} \leq C_2 \|G(t,D_x)w(t,x)\|_{M,T} \tag{6.1}$$

for all w with $Gw \in B_{\vartheta,M,T}$ and all T. Then we fix some positive number K,

$$K := 2C_2 C_{apr}(\|G(0,D_x)u_0(x)\|_{C_{\vartheta,M}} + \|u_1(x)\|_{C_{\vartheta,M}} + 1),$$

where C_{apr} is the constant of Theorem 4.1, and choose $0 < T \leq T_0$ such that

$$T C_{apr} C_1(K) \leq \frac{1}{2C_2}, \tag{6.2}$$

where $C_1(K)$ is from Corollary 5.2. We fix some set $X \subset B_{\vartheta,M,T}$,

$$X = \left\{ w \in B_{\vartheta,M+1/(l_*+1),T} : \|Gw\|_{M,T} + \|w_t\|_{M,T} \leq \frac{K}{C_2} \right\} \subset B_{\vartheta,M,T}.$$

Let $\overline{X}$ denote the closure of X in $B_{\vartheta,M,T}$. If $w, \tilde{w} \in X$, then we have $\|w\|_{M,T} \leq K$, see (6.1); and, consequently, $\|f(t,x,w)\|_{M,T} \leq C_1(K)\|w\|_{M,T}$, $\|f(t,x,w) - f(t,x,\tilde{w})\|_{M,T} \leq C_1(K)\|w - \tilde{w}\|_{M,T}$. By (6.2) and Theorem 4.1, $\mathcal{A}$ maps $\overline{X}$ into X and is a contraction. Then Banach's fixed point theorem gives us a fixed point u of $\mathcal{A}$. □

Finally, we show that the difference $u - v$ has higher regularity than u and v.

Theorem 6.2. *Let u and v be the solutions to* (1.6), (1.7) *and let the assumptions of* Theorem 6.1 *be valid. Then*

$$U - V \in B_{\vartheta,M+1/(l_*+1),T}, \quad u - v \in B_{\vartheta,M+2/(l_*+1),T}.$$

Proof. From $u \in B_{\vartheta,M+1/(l_*+1),T}$ and Corollary 5.2, we deduce that $f(t,x,u) \in B_{\vartheta,M+1/(l_*+1),T}$. The function $u-v$ solves $L(u-v) = f(t,x,u)$ and has vanishing initial data. Then Theorem 4.1 gives the assertion. □

7. An example

Let us illustrate the results of this paper by an example. In [6], the example of Qi Min-You [15] has been extended to Cauchy problems of the type

$$\begin{aligned}&Lv = v_{tt} + 2ct^{l_*}v_{xt} - at^{2l_*}v_{xx} - bl_*t^{l_*-1}v_x = 0, \quad a,b,c \in \mathbb{R},\\ &v(0,x) = u_0(x), \quad v_t(0,x) = 0.\end{aligned}$$

Looking for a solution $v(t,x) = \sum_{k=0}^{m} C_{km}t^{(l_*+1)k}(\partial_x^k u_0)(x + \mu t^{l_*+1})$ we obtain

$$\mu_{1,2} = \frac{1}{l_*+1}\left(-c \pm \sqrt{c^2+a}\right), \quad m_{1,2} = \frac{l_*}{2(l_*+1)}\left(-1 \pm \frac{b+c}{\sqrt{c^2+a}}\right).$$

Assuming that $m_1 \in \mathbb{N}$ ($m_2 \in \mathbb{N}$), we see a loss of m_1 (m_2) derivatives and propagation of singularities along one characteristic only. We find that the loss of regularity given by (4.2), (4.3) and Theorem 4.1 is sharp for this example.

Now suppose that a,b,c are functions of x. We choose two distinct points x_1^0, x_2^0 on the initial line and neighbourhoods V_1, V_2 of x_1^0 and x_2^0. Moreover, assume that a,b,c are constant in V_1, V_2, and that their values are arranged in such a way that $m_1 = m_1(x) \in \mathbb{N}$ for $x \in V_1$ and $m_2 = m_2(x) \in \mathbb{N}$ for $x \in V_2$ with $m_1 \neq m_2$. Then, locally, v has a loss of m_j derivatives in a neighbourhood of x_j^0. This can be seen as follows. The solution to a weakly hyperbolic Cauchy problem is unique if the initial data have high Sobolev smoothness, see [11], [14], [16]. Since unique solutions have finite speed of propagation, the solution v coincides with a function v_j in a neighbourhood of $(x_j^0, 0)$, where v_j solves

$$\begin{aligned}&v_{j,tt} + 2c(x_j^0)t^{l_*}v_{j,xt} - a(x_j^0)t^{2l_*}v_{j,xx} - b(x_j^0)l_*t^{l_*-1}v_{j,x} = 0,\\ &v_j(0,x) = \varphi_j(x)u_0(x), \quad v_{j,t}(0,x) = 0,\end{aligned}$$

with $\operatorname{supp}\varphi_j \subset V_j$, $\varphi_j \equiv 1$ near x_j^0. On the other hand, v_j is given by

$$v_j(t,x) = \sum_{k=0}^{m_j} C_{km_j}t^{(l_*+1)k}(\partial_x^k(\varphi_j u_0))(x + \mu_j t^{l_*+1}).$$

This proves that the solution v suffers from a loss of m_1 derivatives near x_1^0 and of m_2 derivatives near x_2^0, which is exactly the loss predicted in (4.3). Furthermore, we observe that singularities of the datum u_0 in V_1 or V_2 propagate along one characteristic only. Now let u be the solution to $Lu = f(t, x, u)$, $u(0) = u_0$, $u_t(0) = 0$, where f satisfies (1.11). Then Theorem 6.2 states that $u - v$ has higher regularity than v. This means that the strongest singularities of u and v coincide. The function u may have additional singularities produced by nonlinear interaction, but these additional singularities are weaker, at least by the Sobolev order $1/(l_* + 1)$.

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